# A Mean-Reversion Theory of Stock-Market Crashes.

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**Abstract**

Errors in the perception of mean-reversion expectations can cause stock-market crashes. This view was proposed by Fischer Black after the stock-market crash of 1987. I discuss this concept and specify a stock-price model with mean-reversion in returns. Using daily data of the Dow Jones Industrial Average and the S&P500 index I show that mean-reversion in returns is a transient but recurring phenomenon. In the case of the crash of 1987 I show that during the period 1982–1986 mean-reversion was higher than during the nine months prior to the crash. This indicates that mean-reversion expectations were underestimated in 1987. This error was disclosed when in the week prior to the crash it became known that a surprisingly high volume of equities was under portfolio insurance and thus hedged against a faster reversion. Simulations of the model with parameter estimates obtained from the two periods show that a crash of 20 percent or more had a probability of about seven percent. Up to five years after the crash, mean-reversion was higher than before. This supports Black's hypothesis. Contrary to that, the crash of 1929 cannot be explained by a mean-reversion illusion.

# Contents



# 1 Mean-Reversion and Stock-Market Crashes

Mean-reversion is understood in this paper as the change of the market return in the direction of a reversion level as a reaction to a prior change in the market return. After a positive change in the actual returns, mean-reversion causes a negative subsequent change and vice versa. This reverting move can occur with different speeds, it can eliminate the prior change in, say, one day or in one year. Figure 1 illustrates the concept.

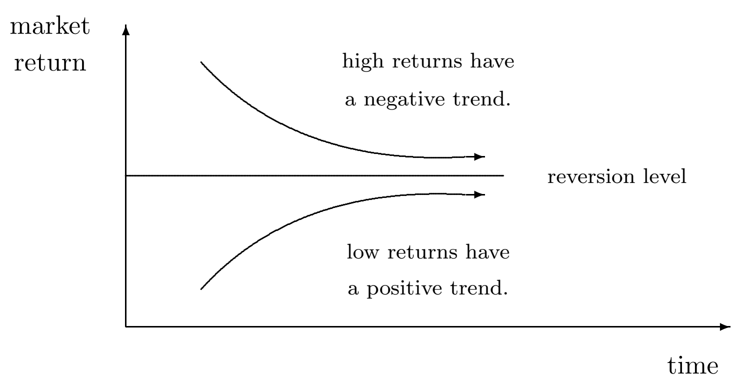


Figure 1: The concept of mean reversion.

In a stock market with mean-reversion in returns, the participants will develop expectations about the speed of the reversion. When a market participant observes for instance a positive change in returns, her reaction to this change will depend on her expectation of the reversion speed. If she has a long position and expects the high returns to disappear quickly, she will probably sell in order to realize the high returns. If she has a short position, she will probably keep this position and cover later when prices are lower. She might even sell more short to gain the difference when prices come down again. If she on the other hand expects the reversion speed to be very slow, then in the case of a long position she will probably hold the paper to get high returns. She might even want to buy when she thinks that more positive moves are possible. In the case of a short position she will probably cover earlier as there is the risk that prices will stay high or even rise. That is, after a positive change the expectation of a fast reversion leads to higher selling pressure than the expectation of a slow reversion.

The mean-reversion expectations of the market participants are not directly observable, they can only be deducted from their sales. High sales after a positive change in returns indicate a fast expected reversion.

Black (1988) proposed that misperceptions in the development of mean reversion expectations can cause stock-market crashes when the participants learn about their error. Black's work was based on a literature that emerged in the late 1980's and discussed the evidence of mean-reversion in stock returns (DeBondt/ Thaler (1985), Summers (1986), Fama/French (1988), Poterba/Summers (1988)).

In this paper, I will propose a stock-price model with mean-reverting returns. Using daily data of the Dow Jones Industrial Average and the S&P500 I will show that there were recurring periods since 1901 where mean-reversion was significant.

Examining the crash of 1987 in detail, I show that it was probably caused by a misperception of mean-reversion expectations as for about five years after the crash, mean-reversion was significantly higher than before the crash. During the period 1982–1986 which was identified as the bull market that led up to the crash by the report of the Brady-Commission, I measure a significantly higher mean-reversion than during the year 1987. This supports the hypothesis that an illusion about the true mean-reversion expectations in the market led to the high price level before the crash. The event that disclosed this illusion can be identified as the surprisingly high volumes of equities under portfolio insurance that became known in the week prior to the crash. Simulations of the proposed model using parameter estimates from the 1982–1986 and the January 1987–October 1987 periods as given by the Brady-Report result in a probability of more than seven percent for a crash of 20 percent or more. A correction of minus 10 percent or more had a probability of over 40 percent.

# 2 A Mean-Reversion Theory of Stock-Market Crashes

## 2.1 Mean-Reversion Expectations

I consider the situation where at time $t$ a positive change is observed (Fig. 2). An individual investor with conservative expectations might now think that returns will come down fast. If $\lambda$ is some parameter in the return generating process that controls the reversion speed, her expectations can be represented by, say, the parameter value $\lambda_0$ which stands for a fast reversion. As the individual investor is not alone on the market, her expectations are probably dependent on the behavior of other participants as well. Let us assume that between times $t$ and $t+h$ she does not act in any way but observes the behavior of the other market participants to come up with an expectation which is some weighted average of her *a priori* expectation indicated by $\lambda_0$ and the observed market behavior. If the market's sales indicate a reversion expectation like the one represented by $\lambda_2$, the investor recognizes that her *a priori* expectation was very conservative relative to the market and consequently adjusts it to $\lambda_1$, for instance.

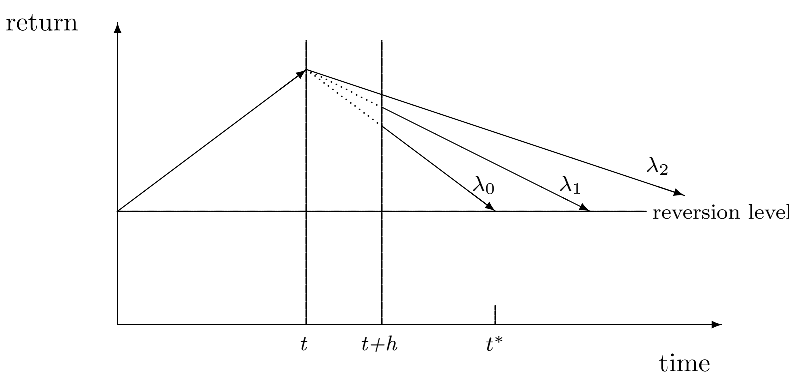


Figure 2: The development of mean-reversion expectations.

The premise is that there are participants who act autonomously, i.e. who do not wait for others to act between times $t$ and $t+h$. These might be institutional investors with predefined investment strategies which, explicitly or implicitly, induce certain reversion expectations. The mean of this implied expectations might be captured by the parameter value $\lambda_2$. Another situation is conceivable. There may be investors who have the same expectation as represented by $\lambda_0$ but who are less risk averse than the individual investor considered. They may follow a strategy which hedges against the case that returns come down faster than according to $\lambda_0$ and at the same time act as if returns would follow $\lambda_2$. This would allow them to participate in gains arising from a slow reversion behavior while at the same time the risk of a faster reversion than $\lambda_0$ would be hedged.

How to implement such a strategy? Each reversion speed $\lambda_{0,1,2}$ corresponds to a certain index (or stock) price at any time. For example, consider time $t^*$ in Fig. 2 as the investment horizon. Let $S(t, \lambda_i)$ denote the index price at time $t$ corresponding to reversion speed $\lambda_i$. Then in $t^*$ I have the relation $S(t^*, \lambda_0) < S(t^*, \lambda_1) < S(t^*, \lambda_2)$ as at that time $\lambda_2$ implies a higher return than $\lambda_1$ than $\lambda_0$, which in turn implies higher respective prices. One alternative would be to buy a put option at strike price $S(t^*, \lambda_0)$ with maturity $t^*$. The investor could control her positions as if she expects the price to behave according to $\lambda_2$. If the price drops below $S(t^*, \lambda_0)$ at time $t^*$, her exposure would be restricted to $S(t^*, \lambda_2) - S(t^*, \lambda_0)$.

The more risk averse individual investor could of course just as well hedge against the possibility that the price falls below her *a priori* level $S(t^*, \lambda_0)$. Her exposure would then be $S(t^*, \lambda_1) - S(t^*, \lambda_0)$ which is less than $S(t^*, \lambda_2) - S(t^*, \lambda_0)$ from which we see that her position is more risk averse.

In the case where a negative change was observed an investor who expects a fast improvement of returns would hold her long positions to avoid realizing temporary losses or buy more to exploit a cost-average effect. She would tend to close short positions to make use of the temporarily low prices. On the other hand, an investor who expects returns to come up slowly or to stay low for a while might want to sell her long positions in order to avoid possibly heavier losses in the future. She would keep or even enlarge short positions to participate from possible further downturns. In summary, after a negative change the expectation of a fast reversion implies higher buying pressure than the expectation of a low reversion.

After a negative change, an investor who wants to hedge against the possibility of a faster reversion while participating from stable low prices could assume short positions as if she expects returns to behave according to $\lambda_2$ (which now means that returns improve slowly). At the same time she could enter into a call option with strike price $S(t^*, \lambda_0)$ and maturity $t^*$. The price relation at time $t^*$ would be $S(t^*, \lambda_0) > S(t^*, \lambda_1) > S(t^*, \lambda_2)$. The risk would be that the price rises quickly, so that the investor would have to cover at higher prices than she got when entering into the short position. If the price at time $t^*$ rises above $S(t^*, \lambda_0)$, her exposure would be restricted to $S(t^*, \lambda_0) - S(t^*, \lambda_2)$.

Again, an individual investor who assumes *a priori* a reversion speed of $\lambda_0$ could wait till time $t + h$ to compare the market behavior. If she sees a reversion speed of $\lambda_2$ she would - just as in the case of a positive change - chose a weighted mean, for instance $\lambda_1$. This would imply lower long and higher short positions than according to $\lambda_0$. If she keeps her suspicion and hedges against prices higher than her *a priori* level $S(t^*, \lambda_0)$, her exposure would be $S(t^*, \lambda_0) - S(t^*, \lambda_1)$ which is lower than $S(t^*, \lambda_0) - S(t^*, \lambda_2)$, the exposure of the investor considered before. Thus, after negative changes as well as after positive changes it is risk-averse to assume a high reversion speed.

## 2.2 Mean-Reversion Illusions and Disillusions

Assume that those investors who are less risk-averse and enter into an option contract while speculating on low reversion speeds give a public record of what they are doing. Then, when the individual investor considered develops her expectations between times $t$ and $t+h$, she will not only look at the market to see how the others play more risky. She will also look at these public records and will recognize that the mean-reversion expectations of those investors who are already active on the market are not that different from her - *a priori* - own but that they have entered into appropriate hedges. Her perception of the market's expected speed of reversion would be higher and thus her own expectation, the weighted average of her *a priori* expectation and her market perception, as well.

The put-call ratio is a proxy for these imaginary public records. If after a positive change in returns investors look at a stable high market to develop mean-reversion expectations, they can conclude from a high put-call ratio that the market's expected mean-reversion speed is higher than indicated by stock sales alone. Conversely, if the market stays low after a negative change, a low put-call ratio indicates the same.

It gets problematic when the risk-tolerant investors choose to synthesize the options contracts. Then they hold hedge portfolios consisting of stocks (or futures) and bonds. It cannot be seen from the buy and sell orders that these transactions are designed to mirror an option and hence there is no record at all. In this case the individual risk-averse investor has no opportunity to infer her expectations from other sources than the stock sales itself. If the market stays high after a positive change or low after a negative change, she will systematically underestimate the market's expected mean-reversion speed. In this case the information that the risk-tolerant investors are not confident of a low reversion speed but hedged against a high one is completely hidden.

How does the crash come about? Assume that the underestimation of the reversion speed is a mass phenomenon and not confined to a single investor because the expectations of the risk-tolerant investors are not or only rudimentary observable. For illustration, consider the extreme case where except for the small group of risk-tolerant investors, all others are more risk-averse. They wait between $t$ and $t+h$ to observe the market without being able to infer the true expectations of the acting investors. The net effect of the market transactions of the risk-tolerant investors accounting for both, their purchases and their short sales from the portfolio that replicates the put option, will be positive after a positive change and negative after a negative change. This assertion is shown in the Appendix.

To stay within the picture of Figure 2 I assume that the $\lambda_2$-position is now the net result of the risk-tolerant group's consolidated purchases and short sales. As the hedging cannot be perceived by the risk-averse investors, who *a priori* assume the speed $\lambda_0$, they adjust their expectations to $\lambda_1$. (Note that every single investor adjusts her expectations without knowing about the others. The move from the mean $\lambda_0$ to the mean $\lambda_1$ is the result of all these adjustments.) We are now able to formulate the mean-reversion speed that will be effective on the market after $t + h$: Denote the proportion of the sales by the risk-averse majority by $\alpha \in [0, 1]$ and the proportion of the sales by the autonomous, risk-tolerant group by $\beta \in [0, 1]$. Then the effective mean-reversion speed will be $\lambda = \alpha\lambda_1 + \beta\lambda_2$. This holds for the illustrative, extreme case that the market consists of these two groups only, i.e. $\alpha + \beta = 1$. The theory set out here is valid as long as $0 < \lambda < \lambda_0$, i.e. the effective reversion speed is lower than the *a priori* reversion speed.

As mentioned in the previous section, it is not necessary for the argument that the risk-averse majority is not willing to enter into options contracts hedging the $\lambda_0$-level. It is sufficient that the mean $\lambda_0$ of *a priori* expectations is larger than $\lambda_2$, the expected reversion speed as implied by the sales of the autonomous group and that this $\lambda_2$ is mistaken to be the true expected reversion speed of the market.

This situation I will call *mean-reversion illusion.* The mean-reversion speed $\lambda$ prevalent on the market is slower than it would have been if the group of risk-averse investors had seen the hedge activity of the autonomous group correctly. Of course this misconception is disclosed if the true expectations of the autonomous group and their hedge positions become known. Then every single investor readjusts her expectations. Yet another disillusion is conceivable: When the majority becomes aware of its majority, that is when it becomes known that a large number of investors had expected a faster reversion but adjusted it to a slower one after observing the activities of a small group. These two disillusions are independent: the information about the true expectations of the autonomous group does not imply that many others followed them. If $\lambda_0$ were observable and the prevalent speed $\lambda$ slower, then this would imply that most participants must have followed some group with seemingly slower expectations, but mostly $\lambda_0$ will be unobservable. Conversely, the information that a majority with high mean-reversion expectations followed a minority with seemingly slow ones does not say anything about the true expectations of the minority.

In the light of this considerations a stock-market crash will be defined as the *mean-reversion disillusion.* If one of the two possible disillusions happens at time $t_c$, it will become clear that the market assumed a false reversion speed since time $t$. That is, the price process followed a 'wrong' trajectory between $t$ and $t_c$. 'Wrong' means that it did not properly reflect the average *a-priori* mean-reversion expectations of the market. This wrong trajectory now has to be eliminated and the process has to be set into a position as if the illusion had not happened.

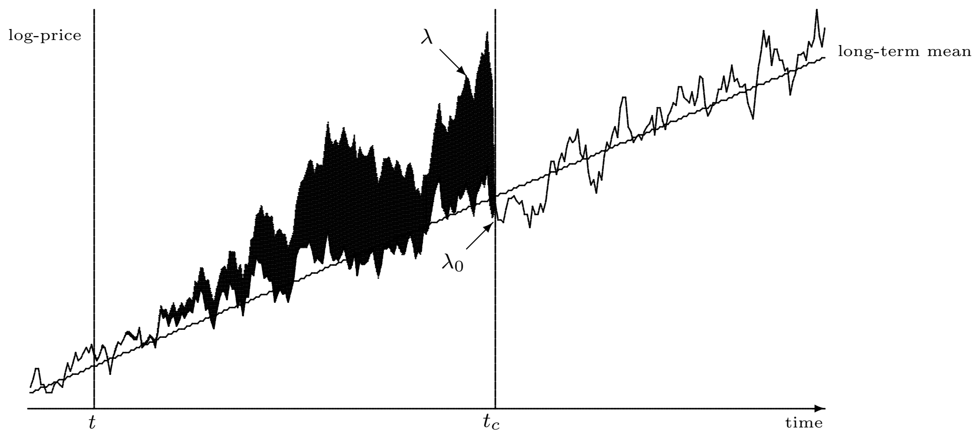


Figure 3: The mean-reversion illusion between times $t$ and $t_c$ and the resulting difference in mean-reversion velocities $\lambda$ and $\lambda_0$ drive the log-price process above the $\lambda_0$-level. The difference in the trajectories is shaded black and gives the potential crash at every point in time.

The crash is thus not just a readjustment in one parameter. Instead, it is this readjustment *plus* a discontinuous correction for the difference in the trajectories induced by $\lambda$ and $\lambda_0$ between $t$ and $t_c$. The precipitousness of the crash depends therefore on the "depth" of the illusion $(\lambda - \lambda_0)$ and its duration $(t_c - t)$. Figure 3 illustrates the point.

The argument is symmetric: It might as well be that during the mean-reversion illusion the price process follows a path below the one given by the higher *a-priori* mean-reversion speed. When the disillusion happens it will cause an upward jump. The magnitude of upward jumps is more restricted than that of downward jumps for a simple economical reason. Most investors have no large pile of money that can be unloaded onto the market in such an instance. They have to shift investments and restructure portfolios which leads to a delay between the decision to buy and the actual purchase. Contrary to that, in the case of a downward jump investors who have decided to sell will be willing to accept cash in any volume.

## 2.3 Mean-Reversion Disillusion and October 19, 1987

Did errors in the perception of mean-reversion expectations play any role in the stock-market crash of 1987? This would mean that there was an illusion and later a disillusion about the market's average *a-priori* mean-reversion expectation. In the notation of Figure 3, I am looking for the points $t$ and $t_c$ and the related events. I do not expect any particular event to cause the illusion at time $t$, so it will be difficult to identify $t$. The point $t_c$ is the point immediately before the crash. The disillusion must be an event or a piece of information that is relevant for mean-reversion expectations and that surprises the public. Following the argument set out in Section 2.2 that a mean-reversion illusion is particularly likely to happen when hedges can be implemented that cannot be recognized by other market-participants, I look for the disclosure of a large hedge position. According to the hypothesis this would imply that a group of active market participants had been more risk-averse than the average investor perceived and that this group was hedged against a high mean-reversion speed.

The three days prior to October 19, 1987, are of prime interest in this respect. From Wednesday, October 14 to Friday, October 16, the U.S. stock market lost more than ten percent. The Dow Jones Industrial Average fell from 2.508 at closing on Tuesday to 2.246 at closing on Friday, the S&P500 from 314 to 282 over the same time. The loss on Wednesday was three percent, on Thursday two percent, and on Friday five percent.

These drops can be attributed to fundamental reasons, namely to the simultaneous budget and trade balance deficit and to the House Ways and Means Committee's plans to eliminate tax benefits for takeovers. On Wednesday, October 14, the U.S. government announced that the trade deficit was about ten percent higher than expected. The dollar fell sharply in reaction, this led to an expected decrease in foreign investment. Also on Wednesday it became known that the Committee actually filed legislation concerning the takeovers (Brady et al. (1988), p. III-2f). Mitchell/Netter (1989) observed that the losses were largely confined to the U.S. market, an indication of the fundamental cause.

Portfolio insurance companies reacted by increasing their cash positions through sales of index futures. They sold 530 million dollars on Wednesday, 965 million dollars on Thursday, and 2.1 billion dollars on Friday, the latter being eleven percent of the total daily sales on the futures market (Brady et al. (1988), p. III-16). At the same time, it became known that these sales were by far not sufficient to adjust the portfolio insurance positions adequately. The report of the Brady-commission that was set in after the crash to determine its causes mentions another eight billion dollars that were expected to be sold on the futures market. It is not clear from the report where these information came from. The implied volume of equities under portfolio insurance, 60 to 90 billion dollars, however seems to have surprised the market. This may have been the event that disclosed the average risk-aversion and the *a-priori* mean-reversion expectations of the market participants (Brady et al. (1988), p. 29).

The Brady-Report and many other authors attributed the cause of the crash partly to the mere existence of portfolio insurance and associated program trading that cascaded in the crash. While this was probably important for the amplification of the downturn, the view proposed here is quite different. The unexpectedly high portfolio insurance volumes were *fundamental* information, not just a technical issue. They revealed that during the boom of 1987 a mean-reversion illusion occurred. This view is closely related to the model of Jacklin/Kleidon/Pfleiderer

(1992): They interpret the high volumes under dynamic hedging as a surprise to the market as well. In this fact they see the fundamental information that a large part of the stock purchases during the boom was not caused by fundamental information but noise. They construct a market model according to this hypothesis and show in simulations that underestimation of portfolio insurance results in a higher market level and that prices fall when the amount of portfolio insurance is revealed. Here, I will specify a stock return model with mean reversion and show that the movements in (actual) mean reversion can indeed be found in the market data.

# 3 A Mean-Reversion Model for Stock Returns

An intuitive way to think about mean reversion in stock prices is to assume that the return process reacts to any deviation from its long-term mean. If the return is above the mean in one period, there is a force that pushes it downwards in following periods, if the return is below the mean, it is pushed upwards.

The mean return induces a certain appropriate stock price, denoted by $\tilde{\vartheta}_t$, which can be interpreted as an estimator of the fundamental value of the underlying stock or stock index. I set

$$\tilde{\vartheta}_t = S_0 e^{\mu\, t}.$$

Consider the return process given by

$$\frac{dS_t}{S_t} = \mu dt + \lambda \frac{\tilde{\vartheta}_t - S_t}{S_t} dt + \sigma dW_t. \tag{1}$$

Here, the magnitude $(\tilde{\vartheta} - S)/S$ measures the deviation of the return process from the long-term mean $\mu$. The parameter $\lambda \geq 0$ controls the speed with which the return is pushed back to the mean $\mu$. The average mean-reversion time is $1/\lambda$ units of time. $W_t$ is standard Brownian Motion. It is shown in the Appendix that the expected value of the process satisfying (1) is

$$\mathbb{E}S_t = S_0 e^{\mu\, t} = \tilde{\vartheta}_t.$$

This is intuitively expected from a mean-reverting process. A similar model was proposed by Metcalf/Hasset (1991).

The process satisfying

$$d \log S_t = \tilde{\mu} dt + \lambda(\log \vartheta_t - \log S_t) dt + \sigma dW_t, \tag{2}$$

where $\tilde{\mu} = \mu - \sigma^2/2$ and $\vartheta_t = S_0 e^{\tilde{\mu} t}$ is a first-order approximation to (1). This is shown in the Appendix. The solution to model (2),

$$\log S_t = \log S_0 + \tilde{\mu}\, t + \sigma \int_0^t e^{-\lambda(t-u)} dW_u, \tag{3}$$

is an Ornstein-Uhlenbeck process. Hence, (2) is a Vasicek-type model for stock returns (Vasicek 1977). The unconditional distribution of the log-price process is given by

$$\log S_t \sim \mathcal{N}\left(\tilde{\mu}\, t + \log S_0, \frac{\sigma^2}{2\lambda}\right), \tag{4}$$

the process is non-stationary. The higher the speed of the mean-reversion $\lambda$ the smaller is the variance as the process will not leave a certain corridor around its mean. Interesting for purposes of time-series analysis is the conditional distribution of the log-returns $\log S_{t+1} - \log S_t$, given the knowledge of the time series through date $t$. It can be read directly from the model (2):

$$(\log S_{t+1} - \log S_t) \sim \mathcal{N}\left(\tilde{\mu} + \lambda(\log \vartheta_t - \log S_t), \sigma^2\right). \tag{5}$$

To estimate the model, I maximize the log-likelihood

$$L(\theta, \{S_t\}_t) = -\frac{T}{2}\log \sigma^2 - \frac{1}{2}\sum_{t=1}^{T}(r_t - \mu - \lambda(\log \vartheta_t - \log S_t))^2. \tag{6}$$

$T$ denotes the number of observations, $\theta = (\mu,\, \lambda,\, \sigma)'$ is the parameter vector, $r_t = \log S_{t+1} - \log S_t$ denotes the logarithmic returns, $\vartheta_t = S_0 e^{\mu\, t}$ as above. I use the 'dfpmin' routine from Press et al. (2002) as well as 'fminunc' routine from the MATLAB optimization toolbox. Both implement a quasi-Newton method with line search using analytical gradients and numerical Hessians. The derivatives are readily calculated from (6).

The unconditional distribution of the log-returns is given by

$$(\log S_{t+1} - \log S_t) \sim \mathcal{N}\left(\tilde{\mu}, \frac{\sigma^2}{2\lambda}(e^{\lambda} - 1) + \frac{\sigma^2}{2\lambda}e^{-2\lambda t}(1 - e^{-\lambda})\right),$$

so that for $t \to \infty$ I obtain the stationary distribution

$$(\log S_{t+1} - \log S_t) \overset{t\to\infty}{\sim} \mathcal{N}\left(\tilde{\mu}, \frac{\sigma^2}{2\lambda}(e^{\lambda} - 1)\right),$$

and thus the maximum likelihood estimates of $\theta = (\mu,\, \lambda,\, \sigma)'$ will be asymptotically normal and the usual statistical inference of the maximum likelihood estimation applies.

This model can be criticized for many reasons. It is not a model of efficient markets, the stock price process (3) is not a martingale. As I am interested in an explanation of stock-market crashes, I accept a model that allows for non-efficiency locally.

It contains two magnitudes that are highly non-trivial to estimate, the mean return $\tilde{\mu}$ (see for example Merton 1980) and the mean-reversion speed $\lambda$. Hence the considered samples must be chosen carefully to make sure that a mean return is estimated that is relevant to the analysis.

The analogon to the Ornstein-Uhlenbeck process in discrete time is the autoregressive process of order one. Our mean-reversion model implies by $\lambda > 0$ that the autoregressive coefficient is negative. Campbell/Lo/MacKinlay (1997) 66f, and Lo and MacKinlay (1988) show that this coefficient is in fact often found to be positive. I will present evidence that the autoregressive parameter when estimated according to the model proposed here is negative. That is, the estimated mean-reversion parameter $\hat{\lambda}$ is positive. Mean-reversion in returns is rarely significant, though.

Also, one might argue that when $\vartheta_t = S_0 e^{\mu t}$ is an estimator of the fundamental value, why should the market trade an asset far above or below this value? In other words, why should a non-negligible distance $\log \vartheta_t - \log S_t$ occur at all in a market with mean-reversion. White (1990) observed for the case of the 1929 stock-market crash that during the boom that preceded the crash, fundamentals were very difficult to evaluate. This was mainly because many companies entered the stock-market that had virtually no dividend history. A similar case can be made for the internet boom at the turn of the century. The quality of an estimator for the fundamental value that uses any type of historical long-term mean is questionable in situations like that. It is of course conceivable to extend the model to capture a higher mean-reversion speed when the distance of the price process to its long-term mean is large. I will use the model (2) for the sake of simplicity.

# 4 Mean-Reversion and the Stock-Market Crash of 1987 in Market Data

The data are daily closings of the Dow Jones Industrial Average ranging from January 2, 1901, to October 2, 2002, covering 27,293 observations. The series was kindly provided by Dow Jones & Company. Also, I use daily closings of the S&P500 ranging from January 4, 1982 to December 30, 1991, covering 2563 observations. The series was obtained from Datastream. All holidays that repeat the price of the previous day were deleted.

Figure 4 shows the logarithmic price series of the Dow Jones over the 101 years that the series covers (upper plot). I estimated model (2) on a rolling window of 250 points length that is moved forward by 10 points every step. The middle plot shows the estimates of $\lambda$, the bottom plot the $t$-statistics of the estimations. The standard errors for this statistic were calculated according to the quasi-maximum likelihood method of White (1982).

The first observation is that all estimates of $\lambda$ are positive. In the light of

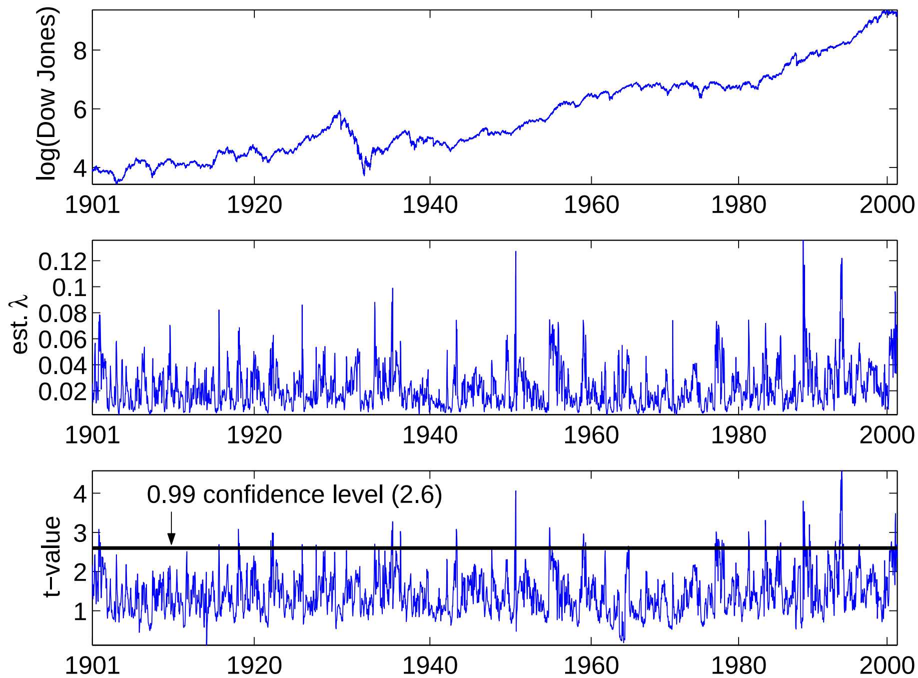


Figure 4: Log-price series of the Dow Jones Industrial Average (top plot), estimations of the mean-reversion speed $\lambda$ according to model (2) on a rolling 250-points window (middle plot), and t-statistics for the estimated mean-reversion speed (bottom plot). The fact that all estimated mean-reversion speeds are positive implies that there is no mean-aversion. Mean-reversion is mostly insignificant but periods occur over the complete sample where it is highly significant.

the findings reported by Lo and MacKinlay (1988) and Campbell/Lo/MacKinlay (1997) 66f, this is a surprising result. It implies that there is no mean-aversion in the daily log-returns of the Dow Jones.

Mean-reversion is mostly insignificant but there are recurring periods over the whole century where mean-reversion is highly significant. Among those are the 1920's and 1930's, the late 1950's, the late 1970's and early 1980's with a clear cluster around the crash, and this year. It must be emphasized that the method used here is based on a moving 250-points mean return. Other concepts of mean returns can lead to different results. Our results are qualitatively insensitive, however, to varying window lengths.

From the considerations of Section 2.3 it is interesting whether there are movements in the mean-reversion parameter occurring around the stock-market crash of 1987 that could be attributed to the mean-reversion illusion and disillusion. I will first look at the disillusion, that is, the crash itself. The hypothesis is that after the crash we should see a faster mean-reversion, that is, a higher $\hat{\lambda}$, than before the crash.

Table 1: Estimation of model (2) on sample periods before and after the 1987 stock-market crash. The observations from October 16, 1987, through October 26, 1987, were deleted from the series. The numbers in parentheses are quasi-maximum-likelihood standard errors according to White (1982). The estimations of the mean returns and standard deviations are significant according to all common confidence levels with the single exception of the mean return of the 100 days before the crash. For the mean-reversion parameter $\lambda$ those estimates that are significant according to the two-sided 0.95 confidence level are marked with a single asterisk, the double asterisk denotes significance according to the two-sided 0.99 confidence level. Mean-reversion speed clearly increased after the crash.

| | **$n$ days before Oct. 16, 1987** | | | **$n$ days after Oct. 26, 1987** | | |
|---|---|---|---|---|---|---|
| $n$ | $\hat{\mu}$ | $\hat{\lambda}$ | $\hat{\sigma}$ | $\hat{\mu}$ | $\hat{\lambda}$ | $\hat{\sigma}$ |
| 100 | 0.000483 | **0.010565** | 0.009632 | 0.001401 | **0.16636**∗∗ | 0.016239 |
| | (0.000613) | (0.011637) | (0.000827) | (0.000114) | (0.061999) | (0.001703) |
| 200 | 0.001172 | **0.005814** | 0.010373 | 0.000851 | **0.077153**∗ | 0.013997 |
| | (0.000401) | (0.005729) | (0.000581) | (7.6e-5) | (0.03205) | (0.001075) |
| 300 | 0.001004 | **0.024993** | 0.010127 | 0.000683 | **0.052885**∗ | 0.012292 |
| | (0.000133) | (0.015156) | (0.000592) | (4.9e-5) | (0.021259) | (0.000859) |
| 400 | 0.000672 | **0.016753** | 0.009907 | 0.000713 | **0.052938**∗∗ | 0.011204 |
| | (0.000110) | (0.008603) | (0.000496) | (3.1e-5) | (0.019714) | (0.000722) |
| 500 | 0.001062 | **0.013118** | 0.009516 | 0.000769 | **0.044117**∗∗ | 0.010935 |
| | (0.000103) | (0.008098) | (0.000425) | (2.9e-5) | (0.014459) | (0.000699) |
| 600 | 0.000834 | **0.015902**∗ | 0.009034 | 0.000715 | **0.033246**∗∗ | 0.01056 |
| | (6.4e-5) | (0.006800) | (0.000379) | (2.8e-5) | (0.011698) | (0.000614) |
| 700 | 0.000951 | **0.017962**∗ | 0.008701 | 0.000663 | **0.023374**∗ | 0.010297 |
| | (4.7e-5) | (0.007320) | (0.000344) | (3.2e-5) | (0.009914) | (0.000554) |
| 800 | 0.000801 | **0.015205**∗ | 0.008523 | 0.000541 | **0.008240** | 0.010536 |
| | (4.3e-5) | (0.0059002) | (0.000314) | (8.1e-5) | (0.005264) | (0.000496) |
| 900 | 0.000728 | **0.010165**∗ | 0.008546 | 0.000563 | **0.009867**∗ | 0.010475 |
| | (5.1e-5) | (0.004539) | (0.000289) | (5.5e-5) | (0.005016) | (0.000451) |
| 1000 | 0.000578 | **0.003738** | 0.008442 | 0.000547 | **0.009902**∗ | 0.010221 |
| | (9.6e-5) | (0.002642) | (0.000269) | (4.5e-5) | (0.004796) | (0.000420) |

## 4.1 The Mean-Reversion Disillusion

First of all, I deleted the observations October 16, 1987, to October 26, 1987, from the returns and the price series of the S&P500 . By this, the crash itself did not affect the estimation of the mean-reversion speed before and after. Then I estimated model (2) for the 100, 200, . . . , 1000 observations before and after the crash. (More precise, before and after the gap.) Table 1 reports the estimations.

The findings clearly support the hypothesis. Up to 700 points before and after the crash, there is an increase in mean-reversion speed. The estimations of the mean-reversion speed $\lambda$ in these samples are significant on the two-sided 0.95 confidence level, four out of seven on the two-sided 0.99 confidence level. As the sample size increase from sample to sample, different mean return concepts are applied here. Except for the 100 and 400 points samples I measure a slightly higher mean return before the crash than after.

As these findings are not independent, I estimated model (2) also on the corresponding opposite intervals of length 200, that is, for the observations *crash-1000* to *crash-800* and *crash+800* to *crash+1000*, then *crash-800* to *crash-600* and

Table 2: Estimation of model (2) on sample periods before and after the 1987 stock-market crash. The observations from October 16, 1987, through October 26, 1987, were deleted from the series. The numbers in parentheses are quasi-maximum-likelihood standard errors according to White (1982). The estimations of the mean returns and standard deviations are significant according to all common confidence levels. For the mean-reversion parameter $\lambda$ those estimates that are significant according to the two-sided 0.95 confidence level are marked with a single asterisk, the double asterisk denotes significance according to the two-sided 0.99 confidence level.

| | $n_i$ | **day $n_i$ through day $n_{i-1}$ before Oct. 16, 1987** | | | **day $n_{i-1}$ through day $n_i$ after Oct. 26, 1987** | | |
|---|---|---|---|---|---|---|---|
| $i$ | $(n_0 = 1)$ | $\hat{\mu}$ | $\hat{\lambda}$ | $\hat{\sigma}$ | $\hat{\mu}$ | $\hat{\lambda}$ | $\hat{\sigma}$ |
| 1 | 200 | 0.001172 | **0.005814** | 0.010373 | 0.000851 | **0.077153**$*$ | 0.013997 |
| | | (0.000401) | (0.005729) | (0.000581) | (7.6e-5) | (0.032050) | (0.001075) |
| 2 | 400 | 0.000672 | **0.016753** | 0.0099065 | 0.000713 | **0.052938**$**$ | 0.011204 |
| | | (0.000110) | (0.008603) | (0.000496) | (3.1e-5) | (0.019714) | (0.000722) |
| 3 | 600 | 0.000833 | **0.015902**$*$ | 0.009034 | 0.000714 | **0.033246**$**$ | 0.010560 |
| | | (6.4e-5) | (0.006800) | (0.000379) | (2.8e-5) | (0.011698) | (0.000614) |
| 4 | 800 | 0.000801 | **0.015205**$*$ | 0.008523 | 0.000541 | **0.008240** | 0.010536 |
| | | (4.3e-5) | (0.005900) | (0.000314) | (8.1e-5) | (0.005264) | (0.000496) |
| 5 | 1000 | 0.000578 | **0.003738** | 0.008442 | 0.000546 | **0.009902**$*$ | 0.010221 |
| | | (9.6e-5) | (0.002642) | (0.000268) | (4.5e-5) | (0.004796) | (0.000420) |

*crash+600* to *crash+800*, and so on. Table 2 reports the estimates. As the mean return concept applied here is a moving 200-days mean, the estimates of the first row are identical to those of the second row of Table 1. The other estimates are not comparable to that of Table 1. With the single exception of the sample corresponding to $n_i = 800$, the estimates support the hypothesis, too.

As the estimates are sensitive to the mean return method, it is interesting to see how they behave when the samples are increased by a finer step-length than 100, as done in Table 1. Again, I delete the days around the crash from the S&P500 series as described above and estimate model (2) on the samples of day *crash* - $n_i$ through *crash - 50*. Then I increase the sample by one day until I estimate (2) on *crash* - $n_i$ through *crash + 50*. The result is an estimation series of length 100. I did these estimations for $n_i = 100, 200, \ldots, 1000$. As a control, I estimated the standard model of geometric Brownian Motion with drift and using this as a null hypothesis, I calculated the likelihood-ratio test statistic.

Figure 5 shows the likelihood-ratio statistic for the estimation series corresponding to $n_i = 100$ (start date May 27, 1987), $n_i = 200$ (start date January 2, 1987), $n_i = 300$ (start date August 11, 1986), and $n_i = 700$ (start date January 8, 1985). For all runs except $n_i = 1000$, the likelihood ratio exceeded the 0.99 confidence level when the sample was increased over the time of the crash. This gives another piece of evidence that mean-reversion significantly increased after the crash. There is no monotonous relationship between the time horizon of the mean return and the significance of the result: The two longest horizons in Figure 5 result in the highest peaks but the shortest, beginning in May 1987, scores

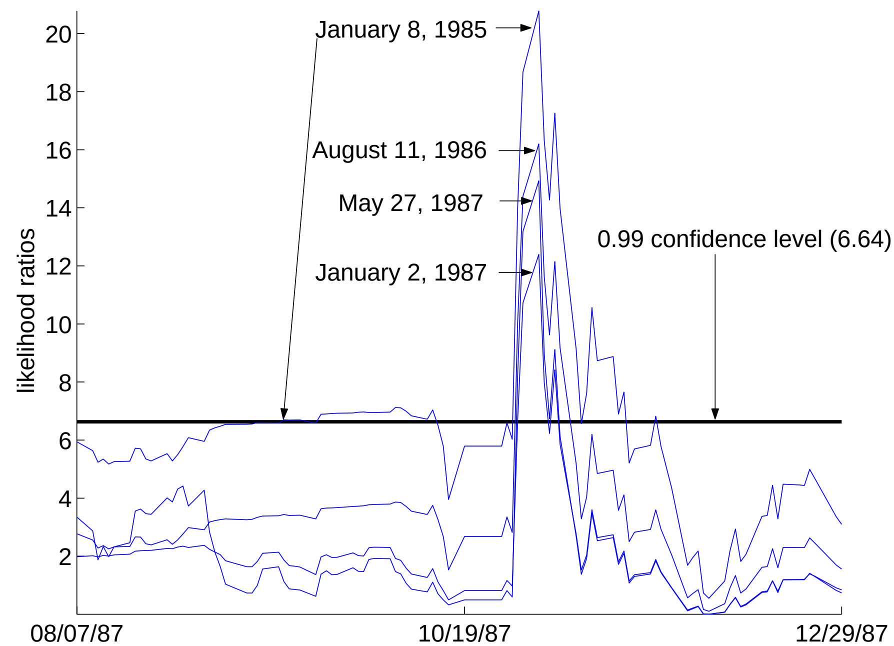


Figure 5: Likelihood-ratio statistics for the alternative hypothesis of (2) against the null hypothesis of geometric Brownian Motion with drift. Model (2) is estimated on the samples of day *crash* - $n_i$ through *crash - 50*. Then I increase the sample by one day until *crash* - $n_i$ through *crash + 50*. The result is a statistics series of length 100, corresponding to the days August 7, 1987, to December 29th, 1987. These estimations were calculated for $n_i = 100, 200, \ldots, 1000$ and the figure shows $n_i = 100$ (start date May 27, 1987), $n_i = 200$ (start date January 2, 1987), $n_i = 300$ (start date August 11, 1986), and $n_i = 700$ (start date January 8, 1985). Except for $n_i = 1000$, all statistics series break through the line of the 0.99 significance level (Chi-square distribution with one degree of freedom: The alternative has one more parameter, $\lambda$, than the null). This shows the increase in the mean-reversion speed after the crash.

higher than the $n_i = 200$ sample starting in January 1987.

## 4.2 The Mean-Reversion Illusion

One of the defining characteristics of the situation of a mean-reversion illusion is that mean-reversion expectations can be implemented without being noticed by the other market participants, for example by synthesized options. Furthermore, the fundamental value of the assets in question is hard to evaluate in this situation. This means that finding the point of the start of the illusion is a much more subtle task than finding the disillusion.

In the notation of Figure 3 I look for the time $t$. That is, I search for a segment of a magnitude of years before the crash where mean-reversion expectations were relatively high. As expectations cannot be measured, I use actual mean-reversion as proxy. According to the hypothesis this segment should be followed by a segment with slower mean-reversion that leads up to the crash.

The Brady-Report locates the beginning of the bull market that led up to the crash in 1982. The contributing factors are described as "*continuing deregulation of the financial markets; tax incentives for equity investing; stock retirements arising from mergers, leveraged buyouts and share repurchase programs; and an increasing tendency to include 'takeover premiums' in the valuation of a large number of stocks*". The valuation of the U.S. stock market by the end of 1986 is described as high but not unprecedented in terms of price/earnings ratios. The appreciation from January 1987 through August 1987, however, "*challenged historical precedent and fundamental justification*" (Brady et al. (1988), p. 9, I-2).

Using this segmentation as a guideline, I estimate model (2) on the segments 01/02/82–12/30/86 and 01/02/87–10/15/87. That is, I set $t =$ January 2, 1987. I assume that the mean return holds for the total period; the model (2) is estimated on the 1987-segment with the mean return set fix at the estimate from the period 1982–1986. Figure 6 illustrates the estimations. The estimate of the mean-reversion speed on the 1982–1986 segment is significant at the one-sided 0.95 significance level. The estimates switch from a higher to a lower value, supporting the hypothesis.

I use a Generalized Likelihood Ratio (GLR) scheme as a changepoint detector (Lai 1995). Let $S = \{S_t\}_{t\in\{1,\ldots,N\}}$ be the considered time series of index prices. The GLR scheme sets a changepoint at

$$\inf_{n\in\{1,\ldots,N\}} \left\{ \max_{1\le k\le n} \sup_{\theta\in\Theta} \left[ \sum_{i=k}^{n} \log \frac{f_\theta(S_i|S_1,\ldots,S_{i-1})}{f_{\theta_0}(S_i|S_1,\ldots,S_{i-1})} \right] > c \right\}, \tag{7}$$

where $N$ is the number of observations and $\Theta$ is the open parameter set. $f_\theta$ is the probability density given the parameter vector $\theta$. $\theta_0$ is the parameter vector of the null hypothesis and $c$ is an *a priori* constant. There is no analytical expression or distribution result for $c$ so that it must be found by simulation methods.

I decomposed the problem (7) into the following steps. On a baseline segment of the first $m$ points of the series I estimated model (2). Thereby I obtain the null hypothesis $\hat\theta_0 = (\hat\mu_0,\, \hat\lambda_0,\, \hat\sigma_0)'$. Then I estimated (2) on every single subseries $\{S_1,\ldots,\, S_j\},\ j = m+1,\ldots,\, N$. This gave us a series of $\hat\theta_j$ maximizing the likelihood functions (6) of the subseries. From this series I computed the probability densities $f_{\hat\theta_j}(S_j|S_1,\ldots,S_{j-1})$ for every $j = m+1,\ldots,\, N$ and stored

$$Z_j := \log \frac{f_{\hat\theta_j}(S_j|S_1,\ldots,S_{j-1})}{f_{\hat\theta_0}(S_j|S_1,\ldots,S_{j-1})}.$$

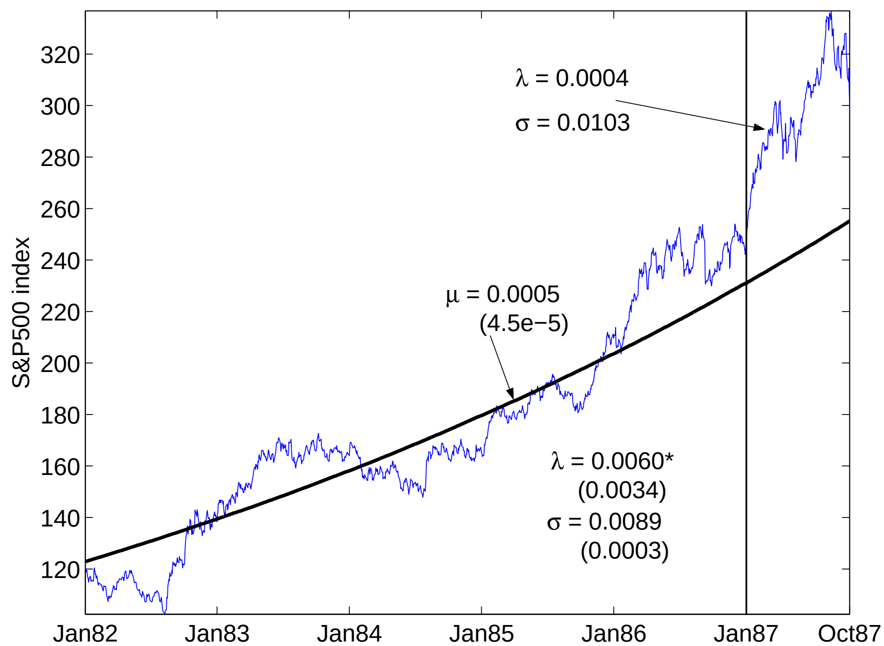


Figure 6: The bull market January 1982 to October 15, 1987 as seen in the S&P500. Using the segmentation of the Brady-Report, I estimate model (2) on the period January 1982 to December 1986 and January 1987 to October 15, 1987. I assume that the same mean return holds for the complete period, it is estimated at 0.0005 from the 1982–1986 segment. The figures in parentheses are standard errors according to White (1982). The estimate of the mean-reversion parameter $\lambda$ on the period 1982–1986 is significant at the one-sided 0.95 significance level. The estimate on the 1987 segment is much lower than the estimate before.

From the resulting series $\{Z_j\}_{j\in\{m+1,\ldots,N\}}$, the statistics series

$$\xi_n = \max_{m+1\leq k\leq n} \sum_{j=k}^{n} Z_j,\ n = m+1, \ldots, N \tag{8}$$

was calculated. As I search for a single changepoint only, it is interesting to plot the $\{\xi_n\}$ series. Figure 7 shows the series when the baseline distribution is estimated on the S&P500 observations January 2, 1982, through December 30, 1985. The series is then calculated for the observations January 2, 1986 through October 15, 1987. It can be seen that the estimated parameters move away from the estimated baseline parameters at two distinct speeds as the sample size increases. This is the interpretation of the two trends in the series that can be distinguished. The trend break is at the turn of the years 1986 to 1987. This supports the observation of the Brady-Report.

A simulation gives the significance levels: I generated 1,000 time series according to model (2) with the parameters obtained from the estimation of the sample

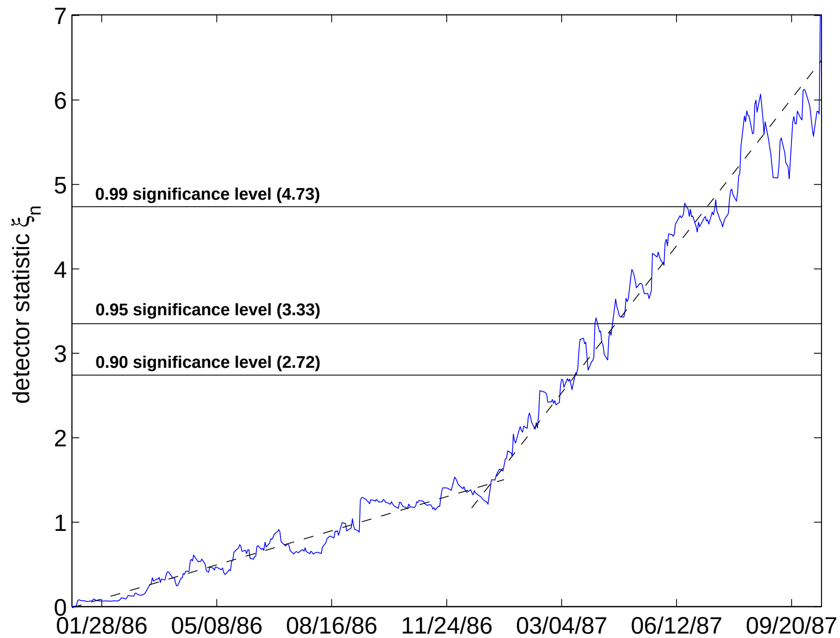


Figure 7: Changepoint detector statistic series $\{\xi_n\}$ as given by Equation (8). The baseline parameter vector $\theta_0$ was estimated on the segment January 2, 1982 through December 30, 1985. The detector statistics series was then calculated for the observations January 2, 1986 through October 15, 1987. Two distinct trends can be observed in the statistic. This means that the estimated parameters move away from the estimated baseline parameters by two distinct speeds as the sample size increases. The trend break is almost exactly at the turn of the years 1986 to 1987, in line with the periods as given by the Brady-Report. The significance levels were obtained by simulation of the statistic.

period January 2, 1982 through December 30, 1985 ($\hat{\mu} = 0.0005$, $\hat{\lambda} = 0.006$, $\hat{\sigma} = 0.009$). This sample consists of 1,012 observations. The sample period January 2, 1986 through October 15, 1987 for which the detector series $\xi_n$ in Figure 7 is depicted consists of 454 observations. Therefore, each of the 1000 simulated time series consisted of 1466 observations. On the first 1,012 observations of each series model (2) is estimated. Then for each series the detector statistic $\xi_n$ is calculated for the remaining 454 observations, yielding 454,000 observations of the detector statistic. The significance levels reported in Figure 7 are the quantiles of these 454,000 observations.

With only this information in hand, what would have been the estimate on October 16, 1987, of the magnitude of a possible crash? More precise, with the information available on October 16, 1987, the question is: Given that the mean-reversion illusion occurred at the beginning of the year 1987, about 200 days ago, and given that the mean-reversion disillusion happens today, what will be the distance in the paths that must be corrected? In the notation of Figure 3 I now look for the distance in the trajectories that is shaded black, measured at the point immediately before the crash. Let me emphasize that I do not estimate the time of the crash, the disillusion is assumed to happen today for whatever reason.

I simulated model (2) with the estimated parameters as reported in Figure 6. I generated 10,000 paths of a random walk of length 200. Then I evaluated model (2) with the parameter vectors obtained from the 1982–1986 segment. The value 246.45 of the S&P500 on January 2, 1987, was set as the starting point. If a mean-reversion illusion occurred in January 1987, it lasted for about 200 days up to October 16, 1987. That is, without the illusion the process would have continued for another 200 days under the old regime. The simulation thus gives an estimate of the distribution of the index value $S_{\text{no illusion}}(200)$ on October 16, 1987, without mean-reversion illusion. The actual value of the S&P500 at the closing of October 15, 1987, was 298.08. I am hence interested in the sample distribution of the difference $\log(S_{\text{no illusion}}(200)) - \log(298.08)$. This is an estimate of the distribution of the magnitude of the crash.

Table 3 (left) shows the sample distribution of the difference $\log(S_{\text{no illusion}}(200)) - \log(298.08)$. There is still a substantial probability for an upward jump as even under the regime with stronger mean-reversion there is a number of paths that end up above 298.08 after 200 days. The probability of a crash of minus 20 percent or more was more than seven percent. The probability of a correction of minus ten percent or more was more than 40 percent.

To put the somewhat random endpoint of 298.08 into perspective, I evaluated model (2) for 10,000 sample paths under both parameter regimes, that of the 1982–1986 period ($S_{\text{no illusion}}$) and that of the 1987 period ($S_{\text{illusion}}$). Table 3 (right) shows the sample distribution of the difference $\log(S_{\text{no illusion}}(200)) - \log(S_{\text{illusion}}(200))$. Even after only 200 days the difference in the mean-reversion parameter $\lambda$ results in substantial distances in the trajectories and thus substantial probabilities for large jumps when a mean-reversion disillusion happens.

These sample distributions were calculated under the assumption that if the mean-reversion illusion had not occurred, the Brownian sample path could have been different from the one that was realized between January 2, 1987, and October 15, 1987. One might argue that the stream of fundamental information that makes up the noise part would have been the same in either case. Under this assumption, I can reconstruct the Brownian sample path between January 2, 1987, and October 15, 1987, from model (2) by

$$\hat{\varepsilon}_t = \frac{1}{\hat{\sigma}} \left[ \hat{\mu} + \hat{\lambda} \log \hat{\vartheta}_t + (1 - \hat{\lambda}) \log S_t - \log S_{t-1} \right]$$

using the parameter estimates from the 1987 segment.

Setting $\hat{\varepsilon}_t$ back in into the model with the parameters from the 1982–1986 segment, this gives a point estimate for the $S_{\text{no illusion}}(200)$ and thus a point estimate

Table 3: The left table shows the sample distribution of the difference $\log(S_{\text{no illusion}}(200)) - \log(298.08)$, the latter value is that of the S&P500 at the close of October 15, 1987. This gives an estimate of the distribution of the magnitude of the crash. The probability of a downward jump of 20 percent or more was more than seven percent. The right table shows the sample distribution of the difference $\log(S_{\text{no illusion}}(200)) - \log(S_{\text{illusion}}(200))$ when 10,000 Brownian sample paths of length 200 are evaluated under both regimes, that of the 1982–1986 period ($S_{\text{no illusion}}$) and that of the 1987 period ($S_{\text{illusion}}$). This shows that the difference in the mean-reversion parameter leads to substantial probabilities for large moves when a mean-reversion disillusion occurs.

| $r_i$ | $\mathbb{P}(r_i - 0.10 \leq r < r_i)$ | $r_i$ | $\mathbb{P}(r_i - 0.10 \leq r < r_i)$ |
|---|---|---|---|
| | | -0.5 | 0.0009 |
| | | -0.4 | 0.0053 |
| -0.3 | 0.0029 | -0.3 | 0.0221 |
| -0.2 | 0.0753 | -0.2 | 0.0751 |
| -0.1 | 0.3652 | -0.1 | 0.1572 |
| 0 | 0.4332 | 0 | 0.2333 |
| 0.1 | 0.1160 | 0.1 | 0.2297 |
| 0.2 | 0.0072 | 0.2 | 0.1687 |
| 0.3 | 0.0001 | 0.3 | 0.0775 |
| | | 0.4 | 0.0244 |
| | | 0.5 | 0.0052 |
| | | 0.6 | 0.0006 |

for the magnitude of the crash. In the case of 1987, I have $S_{\text{no illusion}}(200) = 273.78$ and thus

$$\log(S_{\text{no illusion}}(200)) - \log(298.08) = -0.085,$$

a correction of minus 8.5 percent.

It is conspicuous that the estimated magnitude of the mean-reversion parameter $\lambda$ is much higher after the crash than in the years 1982 to 1986. One reason for this may be that only a part of the mean-reversion expectations after the crash depended on mean-reversion expectations prior to the crash. A general increase in risk-aversion after the crash may have caused an autonomous increase in mean-reversion expectations.

# 5 A Note on the Stock-Market Crash of 1929

The stock-market crash of 1929 can not be explained by a mean-reversion illusion and disillusion. As the knowledge about the hedge-portfolio of the Black-Scholes analysis was not available and option trading was negligible, it was not possible to implement mean-reversion expectations the same way like 1987. An estimation of model (2) in analogy to Table 2 supports this: the result is reported in Table 4. I deleted the observations October 26, 1929, through December 17, 1929, from the Dow Jones series as in this case it took almost two months before the Dow returned to normal daily changes. The results of Table 4 are qualitatively not sensitive to the choice of this gap, I obtained similar results for only ten deleted

Table 4: Estimation of model (2) on sample periods before and after the 1929 stock-market crash. The observations from October 26, 1987, through December 17, 1929, were deleted from the series. The numbers in parentheses are quasi-maximum-likelihood standard errors according to White (1982). The estimations of the mean returns and standard deviations are significant according to all common confidence levels except for the mean return estimation of the 200 days before and after the crash. For the mean-reversion parameter $\lambda$ those estimates that are significant according to the one-sided 0.95 confidence level are marked with a single asterisk. The mean-reversion theory cannot explain the crash of 1929.

| | $n_i$ | **day $n_i$ through day $n_{i-1}$ before Oct. 26, 1929** | | | **day $n_{i-1}$ through day $n_i$ after Dec. 17, 1929** | | |
|---|---|---|---|---|---|---|---|
| $i$ | $(n_0 = 1)$ | $\hat{\mu}$ | $\hat{\lambda}$ | $\hat{\sigma}$ | $\hat{\mu}$ | $\hat{\lambda}$ | $\hat{\sigma}$ |
| 1 | 200 | 9.6e-5 | **0.014254** | 0.013919 | -0.000230 | **0.012919** | 0.015425 |
| | | (0.000546) | (0.014218) | (0.001097) | (0.000550) | (0.010533) | (0.000980) |
| 2 | 400 | 0.001116 | **0.003449** | 0.012798 | -0.001015 | **0.008857** | 0.016447 |
| | | (0.000364) | (0.003398) | (0.000739) | (0.000293) | (0.005738) | (0.000634) |
| 3 | 600 | 0.000990 | **0.009946** | 0.011623 | -0.001654 | **0.004993** | 0.021101 |
| | | (0.000149) | (0.007427) | (0.000558) | (0.000496) | (0.005851) | (0.001057) |
| 4 | 800 | 0.000950 | **0.010573*** | 0.010688 | -0.001663 | **0.008240** | 0.023915 |
| | | (8.3e-5) | (0.006111) | (0.000471) | (0.000273) | (0.003719) | (0.000906) |
| 5 | 1000 | 0.000751 | **0.005945*** | 0.010145 | -0.001415 | **0.005034** | 0.025005 |
| | | (8.6e-5) | (0.003410) | (0.000401) | (0.000224) | (0.003759) | (0.000881) |

days. The change in the mean-reversion parameter $\lambda$ does not have the right sign to support the mean-reversion theory except for the single instance corresponding to $n_i = 400$. The estimates are much less significant, at most at the one-sided 0.95 level and only in two instances *before* the crash. It is conceivable, however that a similar mechanism of error and correction worked for the expected return with coarser instruments like stop-loss orders. I will not pursue this question here, it may be the subject of a separate investigation.

# 6 Conclusions

Errors in the perception of the mean-reversion expectations can cause stock-market crashes. This view was proposed by Black (1988). When the *a-priori* expectation of the speed of the reversion is relatively high but market participants can hedge against a fast reversion and these hedge positions are not public information, a situation may occur that I call *mean reversion illusion* in this paper. That is, a large group $A$ of investors adapts their high *a-priori* mean-reversion expectations to the low expectations that they infer from the market behavior. Investors $A$ do not know that the mean-reversion expectations of those investors $B$ already active on the market are about as high but hedged, for example by synthesized put options that cannot be distinguished from stock sales and purchases due to fundamental information. Investors $A$ adapt their expectations to those that they believe are $B$'s and the stock-price process behaves according to a lower mean reversion speed. When after a while the true *a-priori* expectations of group $B$ become known, for instance because a surprisingly high hedge volume becomes known, the *mean-reversion disillusion* of group $A$ sets in. It is now clear that the stock-price process followed a path that did not properly reflect the true *a-priori* mean-reversion expectations. The process has to be set into a position as if the illusion had not happened. This is a correction in trajectories, not only in process parameters and hence the switch can be of substantial magnitude. This is the stock-market crash.

I specify a stock-price model with mean-reversion in stock returns and estimate it on one hundred years of daily data of the Dow Jones Industrial Average. I show that there are recurring periods where mean-reversion is highly significant. There is no mean-aversion, that is the mean-reversion parameter is always positive.

Using daily data of the S&P500 index I examine the stock-market crash of 1987 in detail. Using the periods of the bull market as proposed by the report of the Brady-Commission, I show that in 1987 mean-reversion was much lower than during the period 1982–1986. This supports the hypothesis of a mean-reversion illusion. Simulations of the model with the estimated parameters of the two segments show that a crash of 20 percent or more had a probability of more than seven percent. A correction of minus 10 percent or more had a probability of more than 40 percent.

There was a significantly higher mean-reversion after the 1987-crash than before. This supports the hypothesis that a mean-reversion disillusion occurred. The cause of the disillusion can be identified as the surprisingly high volumes of equities under portfolio insurance schemes that became known during the week prior to the crash. Not the mere existence of portfolio insurance and cascading program trading caused the crash but the fundamental information that the average *a-priori* mean-reversion expectations in the market were much higher than commonly perceived.

The stock-market crash of 1929 cannot be explained by errors in the perception of mean-reversion expectations. Apart from the fact that synthesized put options were unknown by that time, no significant change in mean-reversion before and after the crash can be measured. The question whether in this case a similar pattern of error and correction concerned the expected return is left for future research.

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**The net effect of market transactions of investors buying a stock and simultaneously replicating a put option on it is positive. That is, the purchases are greater than the sales.**

This will be shown here for the case of a European put option. According to the Black-Scholes model, the replicating portfolio of a European put on one share of the underlying stock consists of a short position of $|\Delta(t)|$. $\Delta$ is the sensitivity of the option to changes in the price of the underlying given by

$$\Delta(t) = \Phi(d_1) - 1 < 0,$$

$$d_1 = \frac{\log \frac{S}{X} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

$\Phi$ is the cumulative distribution function of the standard normal distribution, $S$ is the stock price, $X$ is the exercise price of the put option, $r$ is the risk-free interest rate, $T-t$ is the time to maturity and $\sigma^2$ is the variance of the stock price.

The proceeds from the short position are invested and gain the risk-free interest rate $r$. Assume that the investor hedges every single stock that he buys. His position $P(t)$ then is (in terms of inventories)

$$P(t) = (S - S\Delta(t)e^{rt}) \cdot n,$$

where $n$ denotes the number of shares. The assertion made here is equivalent to

$$\frac{1}{n}P(t) > 0 \quad \Longleftrightarrow \quad S > S\Delta(t)e^{rt}.$$

Now, it is obvious that

$$1 + e^{-rt} > \Phi(d_1),$$

as the exponential function is strictly positive on $\mathbb{R}$ and $\Phi(d_1) \in [0,1]$ as it is a probability. It follows that

$$1 > (\Phi(d_1) - 1)\, e^{rt} \quad \Longrightarrow \quad 1 > \Delta(t)e^{rt}.$$

Multiplying with $S > 0$ proves the assertion.

**The expected value of the process solving model (1) is given by $\vartheta_t = S_0 e^{\mu t}$.**

Rewrite (1) to

$$dS_t = (\mu - \lambda)S_t dt + \lambda\vartheta_t dt + \sigma S_t dW_t,$$

and solve the associated homogeneous equation

$$dX_t = (\mu - \lambda)X_t dt + \sigma X_t dW_t,$$

to obtain $X_t = \exp[(\mu - \lambda - \sigma^2/2)t + \sigma W_t]$. Then the solution to (1) is given by

$$\begin{aligned} S_t &= X_t \left( S_0 + \int_0^t (X_u)^{-1} \lambda \vartheta_u du \right) \\ &= S_0 \exp\left[ \left( \mu - \lambda - \frac{\sigma^2}{2} \right) t + \sigma W_t \right] \left( 1 + \lambda \int_0^t \exp\left[ \left( \lambda + \frac{\sigma^2}{2} \right) u - \sigma W_u \right] du \right) \end{aligned}$$

Taking expectations, I obtain

$$\begin{aligned} \mathbb{E}S_t &= S_0 e^{(\mu - \lambda - \frac{\sigma^2}{2})\,t} \left( \mathbb{E}e^{\sigma W_t} + \lambda \int_0^t e^{(\lambda + \frac{\sigma^2}{2})\,u} \mathbb{E}e^{\sigma(W_t - W_u)} du. \right) \\ &= S_0 e^{\mu t - \lambda t} + S_0 e^{\mu t - \lambda t} \lambda \int_0^t e^{\lambda u} du \\ &= S_0 e^{\mu\, t}. \end{aligned}$$

**Model (2) is a first-order approximation to model (1).**

The mean-reversion term in the model (1) can be rewritten as

$$\lambda \frac{\tilde{\vartheta}_t - S_t}{S_t} dt = \lambda \left( \frac{\tilde{\vartheta}_t}{S_t} - 1 \right) dt.$$

Denote $r := \tilde{\vartheta}_t / S_t - 1$, then

$$1 + r = \frac{\tilde{\vartheta}_t}{S_t}$$

and as $\log(1 + r) \doteq r$ I have a first-order equivalent representation

$$\lambda \frac{\tilde{\vartheta}_t - S_t}{S_t} dt \doteq \lambda \log \frac{\tilde{\vartheta}_t}{S_t} dt = \lambda(\log \tilde{\vartheta}_t - \log S_t) dt.$$

From Ito's Lemma, I have

$$d \log S_t = \frac{dS_t}{S_t} - \frac{\sigma^2}{2} dt.$$

Define $\tilde{\mu} = \mu - \sigma^2/2$ and $\vartheta_t := S_0 \exp(\tilde{\mu}\, t)$. Then there is a first-order equivalent of the model (1) given by (2):

$$\log S_t = \log S_0 + \tilde{\mu}\, t + \lambda \int_0^t (\log \vartheta_u - \log S_u) du + \sigma W_t. \tag{9}$$