Modeling the Recovery Rate in a Reduced Form Model

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(First Version March 9, 2005)

September 15, 2005

Abstract

This paper provides a model for the recovery rate process in a reduced form model. After default, a firm continues to operate, and the recovery rate is determined by the value of the firm's assets relative to its liabilities. The debt recovers a different magnitude depending upon whether or not the firm becomes insolvent. Although this recovery rate process is similar to that used in a structural model, the reduced form approach is maintained by utilizing information reduction in the sense of Guo, Jarrow and Zeng (2005). Our model is able to provide analytic expressions for a firm's default intensity, bankruptcy intensity, and zero-coupon bond prices both before and after default.

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1 Introduction

The credit risk literature studies the valuation and hedging of defaultable financial securities (see Bielecki and Rutkowski (2002), Duffie and Singleton (2003) and Lando (2004) for reviews). In its valuation methodology, two quantities are crucial. The first is the *likelihood of default* (or the *default intensity*, if it exists), and the second is the *recovery rate* in the event of default. The majority of the credit risk research effort involving reduced form models has focused on modelling the stochastic process for the default intensity. Much less work has been done on modelling the recovery rate process itself (one exception is Bakshi, Madan and Zhang (2001)).[1] In these reduced form models, it is usually assumed that the recovery rate is either a constant proportion of the firm's debt value at the instant before default (called the "recovery of market value") or a constant proportion of an otherwise equivalent Treasury's value at default (called the "recovery of face value"), see Bielecki and Rutkowski (2002).

As evidenced by this discussion, the existing reduced form models price risky debt *prior to* default. The pricing of defaulted debt is outside the model's formulation. Yet, markets for defaulted debt exist.[2] Furthermore, these markets are significant in pricing credit derivatives because defaulted debt prices, at default, are the basis for computing recovery rates (see Moody's (2005), Altman, Brady, Resti, Sironi (2003), and Acharya, Bharath, Srinivasan (2004)).[3] With the recent expansion of the credit derivative markets,[4] understanding both defaulted debt prices and the realized recovery rate has become an important issue, especially given the possible introduction of recovery rate swaps (see *Credit Magazine*, "The Next Generation," June 12, 2005).

In light of this gap in the literature, the purpose of the paper is to present a model of the firm's defaulted debt prices and the realized recovery rate in the context of a reduced form model. Taking its insights from the structural

[1] Some structural models also explicitly model the recovery rate process, e.g. Merton [1974].

[2] See Altman, Brady, Resti, Sironi (2003) for a brief discussion of the distressed and defaulted debt markets.

[3] The alternative approach is to measure realized recovery rates, equal to the prices of the debt, but at the emergence of default/bankruptcy, see Moody's (1999) and Acharya, Bharath, Srinivasan (2004).

[4] See Creditflux, issue 38, October 1, 2004 entitled " Surveys confirm rapid market growth."

approach to credit risk, we model the realized recovery rate using the firm's assets and liabilities. But, in order to retain the reduced form structure, we also employ the information reduction methodology of Guo, Jarrow and Zeng (2005). Combined, this generates an extended reduced form model, where the recovery rate process is modelled explicitly in terms of the firm's assets and liabilities. Our approach is consistent with a recent paper by Jarrow and Purnanandam (2004) who study risk management in a model where default, insolvency and bankruptcy are distinct economic conditions of the firm.

Our model can be used to quantify the firm's default process, recovery rate process, and risky debt price, both prior to and after default. These processes are quantified under two information structures related to the firm's asset value: that held by the firm's management (complete information), and that held by the market (partial information). Partial information is characterized by delayed knowledge of the firm's asset value. In our model, default is necessary to trigger the recovery rate process. The recovery rate process, if triggered, depends on the firm's asset value. When in default, if the firm becomes insolvent (defined as the firm's asset value falling below an insolvency barrier) before the debt's maturity, then the firm enters bankruptcy. In bankruptcy, the debt is paid off at some fractional level per dollar owed. In contrast, if the firm remains solvent at the debt's maturity (so that it is not in bankruptcy), then the debt is paid off at a higher fractional level, perhaps unity.

Crucial in understanding the default and recovery rate processes are the default and bankruptcy intensity processes. These characterize the likelihoods of entering default, and within default, of the firm becoming insolvent and entering bankruptcy. Our model shows that when investors have complete information, and if the firm's asset value is below an insolvency threshold at the time of default, then the default and bankruptcy intensities are equal. However, if the asset value is above or equal to this critical level, then the default and bankruptcy intensities are distinct, and default does not necessarily lead to immediate bankruptcy. When investors have partial information, default and bankruptcy are both conceptually and analytically distinct. The formulas for pricing risky debt reflect these information differences and are quantified herein.

Our approach to modelling the recovery rate process generates a pricing formula for *defaulted* risky debt, in contrast to the traditional reduced form credit risk models, whose construction precludes this possibility. In addition, the traditional approach's pre-default debt price will also differ from our

model's price. In the traditional approach, the recovery rate at the time of default is a constant, whereas in our approach, it is a random variable (depending upon a conditional expectation of the realized recovery rate). Properly calibrated, our model should also provide better pre-default prices than the traditional approach.

An outline of the paper is as follows. Section 2 provides the general framework for analysis. Section 3 analyzes the bankruptcy time, while section 4 considers the recovery rate process and risky debt pricing. Section 5 concludes.

2 The General Framework

We consider a reduced form credit risk model for a firm's risky debt. Traded will be a term structure of default free zero-coupon bonds and a firm's risky zero-coupon bond. The firm's risky zero-coupon bond will represent a promised \$1 to be paid at some future date T. If the firm defaults prior to time T, then there will be a recovery rate, between zero and one, paid per promised dollar. Traditional reduced form credit risk models assume that at default: (1) the model ends (default is an absorbing state) and (2) the recovery rate is known. The known recovery rate is either a constant proportion of the debt's value prior to default (called the "recovery of market value") or a constant proportion of an otherwise equivalent Treasury's value at default (called the "recovery of face value"), see Bielecki and Rutkowski (2002). Although convenient, these simplifying assumptions are not satisfied in practice. Firstly, default is not necessarily an absorbing state. Many firms enter default (and even bankruptcy) to emerge later as an operating entity. Secondly, at the time of default, a debt's recovery rate is not known, but represents a random variable, paid at some future (and random) date.

The purpose of this paper is to generalize the traditional reduced form credit risk models by including a stochastic recovery rate process in the event of default. Consistent with this objective, subsequent to default, there will be two economic states of the firm: solvency or insolvency. The firm can remain solvent and pay off the debt as promised[5], or the firm can reach insolvency,

[5] Although this payoff may occur after the debt's maturity date. If this is the case, then the relevant quantity at the debt's maturity is the present value of the debt's promised payment, from the date actually received to the debt's maturity date.

enter bankruptcy, and pay off only a fraction of the debt's value.[6] The notion that default, insolvency and bankruptcy are different states has precedence in the literature. This idea is explicit in Jarrow and Purnanandam (2004) and implicit in Robicheck and Myers (1966), Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997), Mella-Barral (1999), Fan and Sundaresan (2000) and Acharya, Huang, Subrahmanyam and Sundaram (2002).

To present this extension, we must first discuss the default model in its entirety; the task to which we now turn. Our analysis will be based on two representative continuous time stochastic processes for the firm's asset value: a regime switching model with continuous sample paths, and a diffusion model with jumps. We study these processes in order to obtain analytic solutions for the firm's default, bankruptcy and debt price processes. Analytic solutions facilitate intuition, comparative statics and empirical estimation. As discussed below, this structure is readily generalized, but at the expense of losing the analytic solutions.

2.1 The Firm's Asset Value Process

This section presents the two representative processes for the firm's asset value: a regime switching model and a jump diffusion process.

The Regime Switching Model Let $(X_t)_{t\geq 0}$ be the firm's asset value process that follows a diffusion process given by

$$dX_t = X_t \mu_{\epsilon(t)} dt + X_t \sigma_{\epsilon(t)} dW_t \tag{1}$$

where W is a standard one-dimensional Brownian motion, and $(\epsilon(t))_{t\geq 0}$ is a finite-state continuous-time Markov chain, independent of W, taking values $0, 1, \cdots, S-1$ with a known generator $(q_{ij})_{S\times S}$. Moreover, if $\gamma = \inf\{t > 0 : \epsilon(t) \neq \epsilon(0)\}$, then $P(\gamma > t | \epsilon(0) = i) = e^{-q_i t}$, where $q_i = \sum_{j\neq i} q_{ij}$. The drift and volatility coefficients $\mu(\cdot), \sigma(\cdot)$ are functions of ϵ. As indicated, the firm's asset value process evolves continuously in time with drift and volatility coefficients that depend on ϵ, the "state" or the "health" of the firm.

Consistent with Jarrow, Lando and Turnbull (1997), the state space of this continuous time, time-homogenous Markov chain could represent the

[6] In bankruptcy, either liquidation or reorganization occurs, which results in a fractional payoff of the firm's debt obligations.

firm's credit ratings, with $S-1$ being the highest and 1 being the lowest. The last state 0, represents default. Under this interpretation, ϵ represents publicly available information.[7] Note that we do not necessarily require 0 to be an absorbing state. The simplest case is when $S=2$ where $\epsilon(t)=1,0$ correspond to "healthy" and "default," respectively.

In addition, for ease of exposition, ϵ is assumed to be time-homogeneous throughout. As will become clear, the more general case where ϵ is not time-homogeneous is a straightforward extension which adds little extra economic insight.

In the above construction, default is the random time τ given by

$$\tau = \inf\{t > 0 : \epsilon(t) = 0\}.$$

Consistent with the traditional reduced form credit risk models, the default time has an intensity, which is given by $\lambda_t = q_{\epsilon(t)0}$.[8]

The Jump-diffusion Model In the jump-diffusion model, we let W and ϵ be defined as in regime switching model and denote by T_n the n-th jump time of ϵ. We further assign to each state i of ϵ $(0 \leq i \leq S-1)$ a positive random variable ξ_i representing the jump amplitude of the firm's asset value at state i. We assume that $(\xi_i)_{0\leq i\leq S-1}$, ϵ and W are all independent, with $P(\xi_i = 1) = 0$. The firm's asset value process X is assumed to satisfy

$$X_t = X_0 e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t} \prod_{0<s\leq t, \Delta\epsilon(s)\neq 0} \xi_{\epsilon(s)}. \tag{2}$$

Here $\Delta\epsilon(t) := \epsilon(t) - \epsilon(t-)$. Moreover, we assume that ξ_i has a distribution function F_i, and $\Delta F_i(t) := F_i(t) - F_i(t-)$. Since 0 is the default state, we assume $P(\xi_0 \geq 1) = 0$, implying a downward pressure in the firm's asset value while default.

As in the regime switching model, default is the random time τ given by

$$\tau = \inf\{t \geq 0 : \epsilon(t) = 0\},$$

which has an intensity $\lambda_t = q_{\epsilon(t)0}$.

[7] The importance of this statement will become relevant in a subsequent section.

[8] Instead, we could have modelled default as the first hitting time of the asset value to some barrier, a higher barrier than for insolvency (as defined below). This alternative approach is contained in Jarrow and Purnanandam (2004).

2.2 Default, Insolvency, Bankruptcy, and Information

Unlike typical reduced form models, we emphasize the distinction between default, insolvency, and bankruptcy.

Default Default occurs when a firm misses or delays a promised payment on one of its financial liabilities or violates a liability's covenant. Default does not imply that the firm is insolvent (or enters bankruptcy - a legal condition), nor does it imply that all the firm's debt will not pay its promised payments. As mentioned earlier, the publicly available signal concerning the firm's health is represented by the state process $(\epsilon(t))_{t\geq 0}$, and the default time τ is

$$\tau = \inf\{t \geq 0 : \epsilon(t) = 0\}. \tag{3}$$

When default occurs, the firm's asset value process $(X_t)_{t\geq 0}$ changes. In default, the firm faces deadweight losses due to monitoring by the firm's liability holders, 3rd party costs, and possibly the institution of suboptimal investment decisions (see Jarrow and Purnanandam (2004) and references therein). These deadweight losses are reflected in changed drift and volatility coefficients $\mu(\cdot)$ and $\sigma(\cdot)$ in the first model of expression (1) and a sudden downward jump in the firm's asset value in the second model of expression (2).

Insolvency and Bankruptcy In default, there are two possible states of the firm, solvency and insolvency. Consistent with the structural approach to credit risk, once in default, insolvency occurs when the firm's asset value falls to a certain prescribed level x,[9] i.e.,

$$\{X_t < x\}.$$

This insolvency barrier x is inclusive of all deadweight costs incurred immediately at the onset of default.

In default, insolvency induces bankruptcy. Bankruptcy is a legal state of the firm where the shareholders (management) obtain court protection from

[9] One can think of the stopping time as being determined by the firm's assets $\overline{X}_t$ and a stochastic boundary L_t, i.e., $\tau = \inf\{t > 0 : \overline{X}_t \leq L_t\}$. To obtain the characterization in the text, we take (depending upon ease of use) $X(t) \equiv \frac{\overline{X}_t}{L_t}$ or $\equiv \overline{X}_t - L_t$ and we have $\tau = \inf\{t > 0 : \frac{\overline{X}_t}{L_t} \leq 1\}$ or $= \inf\{\overline{X}_t - L_t \leq 0\}$.

the liability holders in order to resolve the firm's financial distress. In our model, insolvency provides the economic rationale for entering bankruptcy. Indeed, when the firm is insolvent, the firm's liabilities cannot be paid in full.

Formally, bankruptcy is defined to be the event

$$\{X_t < x, \epsilon(t) = 0\}.$$

The bankruptcy time is defined by

$$\tilde{\tau} = \inf\{t > 0 : X_t < x, \epsilon(t) = 0\}. \tag{4}$$

Note that if the firm is not in default, then the asset value hitting the barrier x does not induce bankruptcy. This is for two reasons: one, the barrier x reflects the additional deadweight costs paid at the onset of default (these are not incurred if not in default); and two, if not in default, then the creditors cannot force bankruptcy (liquidation and/or reorganization) when the barrier is breached. Furthermore, the firm's asset value process has more favorable drift and volatility coefficients in the non-default state, making eventual insolvency less likely. For future reference, Table 1 contains a convenient characterization of default, insolvency and bankruptcy.

Information (filtration) Structures We study the bankruptcy time under two different filtrations: *complete* and *partial* information. The complete information case corresponds to the information held by management, called investor $\mathcal{A}$. Investor $\mathcal{A}$ sees ϵ, the realization of the state of the firm (including default), and has complete (i.e., continuous) access to the firm's asset value process X. Investor $\mathcal{A}$ could also be interpreted as an informed trader or a government regulator.

The case of partial information corresponds to the information set held by the market, called investor $\mathcal{B}$. This information set determines market prices. Investor $\mathcal{B}$ knows the state of the firm, ϵ, but has limited access to the firm's asset value process X. In particular, we assume that investor $\mathcal{B}$ can only observe the firm's asset value at a times $t_1, t_2, \ldots, t_k, \ldots$, when the firm provides its quarterly report, and at the random times $T_1, T_2, \ldots, T_n, \ldots$, when the state of the firm changes. These random times could be due to the firm issuing press releases or through articles in the financial press. In the jump-diffusion models, additionally observed are the jump sizes of asset value at these random times.

Following Guo, Jarrow and Zeng (2005), it is easy to see that the corresponding filtration (information) structure up to time t for investor $\mathcal{A}$ is the augmented natural filtration of (ϵ, X) such that $\mathcal{F}_t^{\mathcal{A}} = \sigma\{X_s, \epsilon(s), 0 \leq s \leq t\} \vee \mathcal{N}$, where $\mathcal{N}$ is the collection of all negligible sets. In contrast, investor $\mathcal{B}$'s filtration $\mathcal{F}_t^{\mathcal{B}}$ is the augmented minimal filtration generated by the point processes: $1_{\{\tau \leq t\}}$, $\sum_{n=0}^{\infty} X_{t_k} 1_{\{t_k \leq t\}}$, and $\sum_{n=0}^{\infty} X_{T_n} 1_{\{T_n \leq t\}}$ for the regime switching model of expression (1), and $1_{\{\tau \leq t\}}$, $\sum_{n=0}^{\infty} X_{t_k} 1_{\{t_k \leq t\}}$, $\sum_{n=0}^{\infty} \xi_{\epsilon(T_n)} 1_{\{T_n \leq t\}}$ and $\sum_{n=0}^{\infty} X_{T_n} 1_{\{T_n \leq t\}}$ for the jump diffusion model of expression (2).

3 The Bankruptcy Time

The first step in understanding the firm's recovery rate process is to characterize the bankruptcy time under the different information sets held by investors $\mathcal{A}$ and $\mathcal{B}$. The techniques used in this paper for characterization of the bankruptcy time are extensions of those developed in Guo, Jarrow and Zeng (2005).

3.1 The Regime Switching Model

To simplify the presentation and without loss of generality, we assume that the firm starts from a non-default state, i.e., $\epsilon(0) \neq 0$.

Investor $\mathcal{A}$ (Management) Recall that our bankruptcy time $\tilde{\tau} = \inf\{t > 0 : X_t < x,\ \epsilon(t) = 0\ \}$. Under investor $\mathcal{A}$'s information set, if default happens (i.e., $\epsilon(t) = 0$) and if the value of X_t is known, then insolvency (and hence bankruptcy) is dictated by the level of X_t. More precisely, because of the continuous sample path of X, in the next instantaneous time interval $(t, t + \Delta t)$, the bankruptcy intensity is nonzero if and only if $X_t < x$. Moreover, if $X_t < x$, then default immediately leads to bankruptcy, and the bankruptcy intensity equals the default intensity $q_{\epsilon(t)0}$. However, when $X_t \geq x$, the default and bankruptcy intensities are different. In this case the bankruptcy intensity is less than the default intensity $q_{\epsilon(t)0}$ because there is a positive probability that bankruptcy can be avoided.

According to Meyer's previsibility theorem, (see Rogers and Williams (2000)), $\tilde{\tau} = \min(\tilde{\tau}_\Lambda, \tilde{\tau}_{\Lambda^c})$ has a totally inaccessible component $\tilde{\tau}_\Lambda$ and a

totally accessible component $\tilde{\tau}_{\Lambda^c}$ (see Appendix B for more details on the decomposition of a general stopping time into its predictable and totally inaccessible components).[10] The next theorem characterizes this decomposition.

Theorem 1 *Assume expression (1). The bankruptcy time $\tilde{\tau}$ for investor $\mathcal{A}$ has a totally inaccessible component $\tilde{\tau}_\Lambda$, with intensity $d_t^{R,\mathcal{A}}$, given by*

$$d_t^{R,\mathcal{A}} = 1_{\{\tilde{\tau}>t,\ \epsilon(t)\neq 0\}} q_{\epsilon(t)0}(1_{\{X_t<x\}} + \frac{1}{2}1_{\{X_t=x\}}). \tag{5}$$

The above theorem has an intuitive interpretation. Viewing the bankruptcy time $\tilde{\tau}$ as the first hitting time of the set $(-\infty, x) \times \{0\}$ for the joint process $(X_t, \epsilon(t))$, then at any given time $t < \tilde{\tau}$, in order for $\tilde{\tau} \in (t, t+\Delta t)$, there are two possibilities: (1) $X_t \geq x, \epsilon_t = 0$, and (2) $X_t < x$ but with $\epsilon(t) = j \neq 0$. For case (1), with probability $1 - q_{00}\Delta t \sim 1$, $\epsilon(t)$ remains at 0 and in the meantime $X(\cdot) \downarrow x$ in a continuous fashion. In this case the first hitting time is realized in a predictable fashion. This predictable component does not contribute to the bankruptcy intensity of $\tilde{\tau}$. Case (2) generates $\tilde{\tau}_\Lambda$. Here the bankruptcy arrival is due to a sudden jump to $\epsilon(s) = 0$ for some $s \in (t, t+\Delta t)$. This jump time is totally inaccessible, with a rate of q_{j0} equal to the default arrival rate. This bankruptcy rate is halved when $X_t = x$ (where it is the same instantaneous probability for $X_s < x$ as it is for $X_s > x$), see Appendix A for details.

Investor $\mathcal{B}$ (The Market) In contrast to the case for investor $\mathcal{A}$ where the default time equals the bankruptcy time if $X_t < x$, the bankruptcy time is conceptually and analytically different from the default time under partial information. First, the filtration $\mathcal{F}^{\mathcal{B}}$ is of a "delayed" type as in the general framework of Guo, Jarrow and Zeng (2005), and delayed information induces an intensity for bankruptcy. This intensity, denoted as $d_t^{R,\mathcal{B}}$, can be derived via calculating

$$d_t^{R,\mathcal{B}} = \lim_{h\downarrow 0} \frac{P(t+h \geq \tilde{\tau} > t | \mathcal{F}_t^{\mathcal{B}})}{h}$$

(see Appendix C for details).

[10] The stopping time $\tau_\Lambda \equiv \begin{cases} \tau \text{ if } \omega \in \Lambda \\ \infty \text{ if } \omega \in \Lambda^c \end{cases}$, see Rogers and Williams (2000, p. 11).

To understand this intensity process, recall that investor $\mathcal{B}$ is provided at (current) time t with delayed information about the firm's asset value $X_{t_k \vee T_n}$ where t is assumed to satisfy $t_k \leq t < t_{k+1}$, $T_n \leq t < T_{n+1}$. Now, suppose that default occurs at time t and the most updated information about the firm's asset value is from time $t_k \vee T_n$ with $X_{t_k \vee T_n} < x$. Then, when $t = t_k \vee T_n$, clearly default triggers bankruptcy. However, if $t_k \vee T_n < t$, then with this "delayed" nature of information about X, default at time t does not necessarily lead to immediate bankruptcy. Indeed, there is a positive probability that bankruptcy is avoided at time t because the firm's asset value may have moved from below x at time $t_k \vee T_n$ to above x at (current time) t. On the other hand, even if default has not yet happened at time t, the fluctuation of X and the randomness of ϵ can lead to default and afterwards bankruptcy, but with less probability.

To formalize this intuition, note that given the structure of the Markov chain ϵ, the probability of ϵ changing more than two states between t and $t + \Delta t$ is of order $(\Delta t)^2$. Therefore, in default, the probability of avoiding insolvency (and hence bankruptcy) can be easily computed via calculating the probability of the running minimum of X during $(t, t+\Delta t)$ staying above x given $\epsilon(t)$ and $X_{t_k \vee T_n}$. That is,

Theorem 2 *Assume expression (1). The bankruptcy time $\tilde{\tau}$ for investor $\mathcal{B}$ is totally inaccessible under filtration $\mathcal{F}_t(\tilde{\tau})$. Moreover, if $t \in [t_k, t_{k+1})$, then when $\tilde{\tau} > t$, the bankruptcy intensity is*

$$d_t^{R,\mathcal{B}} = \begin{cases} -\frac{\psi_t(\theta_0, t-t_k \vee T_n, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})}{\psi(\theta_0, t-t_k \vee T_n, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})}, & \text{if } \epsilon(t) = 0, \\ q_{\epsilon(t)0} \Phi\left(\frac{\frac{1}{\sigma_{\epsilon(t)}} \ln \frac{x}{X_{t_k \vee T_n}} - \theta_{\epsilon(t)}(t - T_n \vee t_k)}{\sqrt{t - T_n \vee t_k}} \right), & \text{if } \epsilon(t) \neq 0, \end{cases}$$

where $\theta_i = \frac{\mu_i}{\sigma_i} - \frac{\sigma_i}{2}$, Φ is the distribution function of standard normal random variable, and

$$\psi(\theta, t, y) = P(\inf_{0 \leq s \leq t} W_s^{(\theta)} > y) = 1 - \int_0^t \frac{|y|}{\sqrt{2\pi s^3}} e^{-\frac{(y-\theta s)^2}{2s}} ds \quad \text{for } y < 0,$$

with $W_t^{(\theta)} := W_t + \theta t$. ψ_t is the derivative of ψ w.r.t. the variable t.

The detailed derivations are in Appendix C.

3.2 The Jump Diffusion Model

The methodology utilized in the previous section can be similarly applied to the jump diffusion model where the firm's asset value process X follows expression (2). We therefore summarize the main results here without repeating the proofs.

Investor $\mathcal{A}$ (Management) Again, according to Meyer's previsibility theorem $\tilde{\tau} = \min(\tilde{\tau}_\Lambda, \tilde{\tau}_{\Lambda^c})$ has a totally inaccessible component $\tilde{\tau}_\Lambda$ and totally accessible component $\tilde{\tau}_{\Lambda^c}$.

Theorem 3 *Assume expression (2). The totally inaccessible component $\tilde{\tau}_\Lambda$ has an intensity, denoted as $d_t^{J,\mathcal{A}}$. Let $\{\tilde{\tau} > t, \epsilon(t) \neq 0\}$, then*

$$d_t^{J,\mathcal{A}} = q_{\epsilon(t)0} \left[F_0(\frac{x}{X_t}-) + \frac{1}{2}\Delta F_0(\frac{x}{X_t}) \right]$$

where F_0 is the distribution function of ξ_0 and $\Delta F_0(t) := F_0(t) - F_0(t-)$.

The intuition for this theorem is similar to that for Theorem 1, except that here X_t is no longer a continuous process and therefore its jump component also contributes to the bankruptcy intensity. In fact, comparing the bankruptcy intensity $d_t^{J,\mathcal{A}}$ of the jump diffusion model with that of the regime switching model $d_t^{R,\mathcal{A}}$, we see that $d_t^{J,\mathcal{A}}$ consists of both the default arrival rate $q_{\epsilon(t)0}$ and the jump component $F_0(\frac{x}{X_t}-)$ from the diffusion process, while $d_t^{R,\mathcal{A}}$ only equals the default intensity $q_{\epsilon(t)0}$ because of the continuous sample path. In both cases, however, the nature of the bankruptcy "surprise" is a direct consequence of the discontinuity (or, the "exogenous" nature) of the underlying process, either from $\epsilon(t)$ or from the jump component of the asset process.

Investor $\mathcal{B}$ (The Market) Recall that in this case we are given a sequence of constant times $\{t_k\}_{k\geq 0}$. For any fixed time t, if $t_k \leq t < t_{k+1}$ and $T_n \leq t < T_{n+1}$, observed are $X_{t_1}, \cdots, X_{t_k}, T_1, \cdots, T_n, \xi_{\epsilon(T_1)}, \cdots, \xi_{\epsilon(T_n)}, X_{T_1}, \cdots,$ and X_{T_n}. This is equivalent to observing $W_{t_1}, \cdots, W_{t_k}, T_1, \cdots, T_n, \xi_{\epsilon(T_1)}, \cdots, \xi_{\epsilon(T_n)}, W_{T_1}, \cdots,$ and W_{T_n}.

Theorem 4 *Assuming expression (2) and* $\{\tilde{\tau} > t, t_k \leq t < t_{k+1}, T_n \leq t < T_{n+1}\}$*, the bankruptcy intensity, denoted as* $d_t^{J,\mathcal{B}}$*, is given by*

$$\begin{aligned} d_t^{J,\mathcal{B}} &= -\frac{\psi_t(\theta, t - t_k \vee T_n, \frac{1}{\sigma}\log\frac{x}{X_{t_k \vee T_n}})}{\psi(\theta, t - t_k \vee T_n, \frac{1}{\sigma}\log\frac{x}{X_{t_k \vee T_n}})}, \qquad \epsilon(t) = 0, \\ d_t^{J,\mathcal{B}} &= q_{\epsilon(t)0} P(y\xi_0 e^{\sigma W_u^{(\theta)}} \leq x)|_{u=t-t_k\vee T_n, y=X_{t_k\vee T_n}} \\ &= q_{\epsilon(t)0}\int_0^1 F_0(dv)\Phi\left(\frac{\frac{1}{\sigma}\log\frac{x}{vX_{t_k\vee T_n}} - \theta(t - t_k \vee T_n)}{\sqrt{t - t_k \vee T_n}}\right), \qquad \epsilon(t) \neq 0. \end{aligned}$$

The above formula for the bankruptcy intensity illustrates how bankruptcy occurs. Given the delayed information of the firm's asset value at $t_k \vee T_n$, suppose that the firm is in a non-default state $\epsilon(t) \neq 0$. Here, in order to become insolvent, the firm's asset value X_t must drop below level x at time t from $X_{t_k \vee T_n}$ and default $\epsilon(t)$ must occur, with intensity $q_{\epsilon(t)0}$. The bankruptcy intensity, thus, comes from both the change in ϵ and the jump of the firm's asset value process: it is the arrival intensity of the default state times the probability that firm's asset value jumps across the insolvency barrier. In contrast, after default $\epsilon(t) = 0$, the bankruptcy intensity is only due to the jump of the firm's asset value process.

If the firm has not defaulted $\epsilon(t) \neq 0$ and if F_0 is continuous, then it is interesting to note that due to delayed information, the bankruptcy intensity for investor $\mathcal{B}$ is smaller than that for investor $\mathcal{A}$: although both investors $\mathcal{A}$ and $\mathcal{B}$ know that the last observed asset value was below level x (at time $t_k \vee T_n$ and t respectively), for investor $\mathcal{B}$ the asset value has a positive probability of moving above x between time $t_k \vee T_n$ and t and has more chance of solvency. Of course, as $t \to t_k \vee T_n$, the bankruptcy intensity for investor $\mathcal{B}$ converges to that of $\mathcal{A}$.

It is important to point out that the bankruptcy intensity $d_t^{J,\mathcal{B}}$ depends on the typical independent variables used in empirical hazard rate estimation procedures for bankruptcy (see Chava and Jarrow (2004) and references therein). More precisely, as indicated in the explicit representation above, the bankruptcy intensity depends on the firm's health $\epsilon(t)$, the drift of the log(asset) price process $\mu_{\epsilon(t)} - \frac{\sigma^2_{\epsilon(t)}}{2}$, the volatility of the log(asset) price process $\sigma_{\epsilon(t)}$, and the firm's debt/asset value ratio $\frac{x}{X_t}$. As expected, as the state of the firm changes from healthy $\epsilon(t) \neq 0$ to default $\epsilon(t) = 0$, the bankruptcy intensity increases. As the drift of the asset price process $\mu_{\epsilon(t)} - \frac{\sigma^2_{\epsilon(t)}}{2}$ increases, the

intensity decreases. As the volatility of the asset price process $\sigma_{\epsilon(t)}$ increases, the intensity increases. Finally, as the firm's debt/asset ratio increases, the firm's bankruptcy intensity also increases. All these comparative statics are as expected.

4 The Recovery Rate Process and Risky Debt Pricing

We next consider the recovery rate process (after default) and the pricing of the firm's risky zero-coupon bonds for both investors $\mathcal{A}$ and $\mathcal{B}$. With the introduction of the bankruptcy time and hence the differentiation between bankruptcy and default, it is natural to consider the probability of bankruptcy given default. This distinguishes our approach from the traditional risky debt pricing methodology.

For pricing purposes, we assume the existence of an equivalent martingale measure making the discounted risky zero-coupon bond's price a martingale. This is equivalent to assuming an arbitrage free market (see Bielecki and Rutkowski (2002) for details). In incomplete markets, defined by the regime-switching and jump diffusion models, it is well-known that such an martingale measure exists, but it is not unique (see Guo, Jarrow and Zeng (2005) for discussions on this issue). For the subsequent analysis, we fix a particular measure from this set of equivalent martingale measures, assuming that the market is in equilibrium. For simplicity of notation (and the exposition), we let the probability measure P underlying the regime switching model (expression (1)) and the jump-diffusion model (expression (2)) be this martingale measure.

4.1 Review: The Traditional Approach

For easy comparison, we shall briefly review the traditional approach to pricing risky debt. Consider a zero-coupon bond issued by the firm paying \$1 at time T if there is no default, and $\$\widetilde{R}$ at time T if the firm defaults prior to time T. For simplicity of exposition, we let $\widetilde{R} \in [0,1]$ be a constant, although it is possible to extend the analysis to a $\mathcal{F}_\tau^i$ measurable random variable $\widetilde{R}$. This formulation is called the *face value of debt recovery rate* process (see Jarrow and Turnbull (1995)). Similar results hold for other independent re-

covery rate processes (see Bielecki and Rutkowski (2002) for the relevant alternatives).

Given this structure, assuming that the interest rate process is deterministic, the value of the firm's zero-coupon bond to either investor $\mathcal{A}$ or $\mathcal{B}$ can be written as:

$$\begin{aligned} V_C^i(t,T) &= e^{-\int_t^T r(s)ds} E[\widetilde{R} 1_{\{\tau \leq T\}} + 1_{\{\tau > T\}} \mid \mathcal{F}_t^i] \\ &= e^{-\int_t^T r(s)ds} [1 - (1 - \widetilde{R}) P(\tau \leq T \mid \mathcal{F}_t^i)] \text{ for } t \leq \tau \end{aligned}$$

with $i \in \{\mathcal{A}, \mathcal{B}\}$. As indicated, the traditional approach prices the firm's risky debt prior to (or at) default. Note that the difference in prices between investor $\mathcal{A}$ and $\mathcal{B}$ is quantified by the difference between the conditional probabilities of default before time T for investors $\mathcal{A}$ and $\mathcal{B}$. The next proposition characterizes the relationship between these conditional probabilities.

In general Markov models, Guo, Jarrow and Zeng (2005) showed that there is a non-linear relation between the default probabilities under different filtration structures. (This is a generalization of similar results by Collin-Dufresne, Goldstein and Helwege (2003) in the Brownian motion model and by Jeanblanc and Valchev (2004)).

More precisely,

Proposition 1 *Let $D^{\mathcal{A}}(t,T)$ and $D^{\mathcal{B}}(t,T)$ denote the conditional probability of default before time T under the natural filtration $(\mathcal{F}_t^{\mathcal{A}})_{t \geq 0}$ and the delayed filtration $(\mathcal{F}_t^{\mathcal{B}})_{t \geq 0}$, respectively. Let τ_0 be a general stopping time. Suppose at time t, on the event $\{\tau_0 > t\}$, $\sigma(X_s) \subset \mathcal{F}_t^{\mathcal{B}} \subset \mathcal{F}_s^{\mathcal{A}}$ where $s < t$ and $(X_t)_{t \geq 0}$ is the underlying Markov process, then*

$$D^{\mathcal{B}}(t,T) 1_{\{\tau_0 > t\}} = \frac{D^{\mathcal{A}}(s,T)}{D^{\mathcal{A}}(s,t)} 1_{\{\tau_0 > t\}}. \tag{6}$$

Recall that $D^{\mathcal{B}}(0,T) = D^{\mathcal{A}}(0,T)$, because at time 0 both investors $\mathcal{A}$ and $\mathcal{B}$ have the same information. Moreover, when $s \uparrow t$, $D^{\mathcal{B}}(s,T)$ and $D^{\mathcal{A}}(s,T)$ converge to the same value. This relationship is independent of the risk-neutral measure under consideration, as long as the price process X remains Markovian under the pricing measure (This is true under quite general conditions, see Dynkin (1965) page 306 and Palmowski and Rolski (2002)). This shows that "less" information can be as good as complete information as long as it is updated. Finally, one does not necessarily have $D^{\mathcal{A}}(t,T) > D^{\mathcal{B}}(t,T)$ or $D^{\mathcal{A}}(t,T) < D^{\mathcal{B}}(t,T)$ because less information does not necessarily mean the conditional default probability is larger.

Remark 1 *It is worth noting that Proposition 1 and expression (6) apply to any general stopping time τ_0, including default times and bankruptcy times, under the change of filtrations.*

4.2 The Recovery Rate Process

This section presents the recovery rate process for risky debt under our extension. Considering the risky zero-coupon bond with maturity T, we assume that:

- the firm pays \$1 at time T if there is (and has been) no default,
- once default occurs ($\tau < T$), the firm either becomes insolvent and bankruptcy occurs, i.e., $\tau \leq \tilde{\tau} \leq T$; or stays solvent until T, i.e., $\tilde{\tau} > T$,
- if bankruptcy occurs before the debt's maturity T, then the bond pays a *realized* recovery rate of $\$R$, and
- if default occurs and the firm remains solvent up to the debt's maturity T, then the bond pays a fractional recovery rate of $\$K$ where $R \leq K \leq 1$.

Again, for simplicity, we assume that both K and R are constants, although the analysis is easily extended for an $\mathcal{F}^i_{\tilde{\tau}}$ measurable random variable K and an $\mathcal{F}^i_T$ measurable random variable R.[11] Note that because immediate bankruptcy is worse than bankruptcy at a later date, $R \leq K$. For easy reference, the recovery rate process is summarized in Table 1.

This recovery rate process is similar to that used in the structural approach to credit risk with a default barrier. The imposition of an exogenous recovery rate (a constant) if the barrier is breached accounts for typical violations of the creditor's absolute priority rules (see Jarrow and Turnbull (1995, page 58) for related discussions).

Estimates for the realized recovery rate R after default can be found in Moody's (1999) and Acharya, Bharath, Srinivasan (2004) where the realized recovery rate is estimated as the price of the defaulted debt after emergence

[11] There is some evidence that R is independent of the debt's maturity and coupon structure, and only depends on the debt's senority, see Acharya, Bharath, Srinivasan (2004, footnote 10).

from financial distress. The fractional recovery rate K corresponds to the price of the defaulted debt at its original maturity, and (usually) before resolution of the firm's financial distress. Unfortunately, direct estimates for K are not readily available.[12] However, given our model, implicit estimates of K can be obtained. This implicit estimation procedure will be discussed in a subsequent section.

Given this new structure, we have

Proposition 2 *The time t value of the firm's zero-coupon bond to investor $\mathcal{A}$ or $\mathcal{B}$ is*

$$V^i(t,T) = e^{-\int_t^T r(s)ds} E_t^i \left[1 \cdot 1_{\{\tau > T\}} + K \cdot 1_{\{\tilde{\tau} > T, \ \ \tau \leq T\}} + R \cdot 1_{\{\tilde{\tau} \leq T, \ \ \tau \leq T\}}\right]$$

where $E_t^i = E[\cdot|\mathcal{F}_t^i]$ is under a martingale measure with $i \in \{\mathcal{A}, \mathcal{B}\}$.

If at time t, default has not happened, i.e., $\tau > t$, then

$$\begin{aligned} V^i(t,T) &= e^{-\int_t^T r(s)ds}[E_t^i[1_{\{\tau > T\}}] + (K-R)E_t^i[1_{\{\tilde{\tau} > T, \tau \leq T\}}] \\ &\quad + RE_t^i[1_{\{\tau \leq T\}}]]. \end{aligned}$$

If defaulted by time t, but still solvent, then assuming the default state is absorbing,

$$V^i(t,T) = e^{-\int_t^T r(s)ds}[(K-R)P(\inf_{t \leq v \leq T} X_v > x| \ \mathcal{F}_t^i) + R].$$

This later expression is useful for pricing *defaulted* debt in secondary market trading. The traditional approach is not formulated for this situation.

The key quantities to evaluate in the above expressions are the distributions of the default/bankruptcy times for regime switching and jump diffusion models. In some cases, closed-form analytical expressions are available.

4.2.1 Example: Jarrow-Lando-Turnbull

A special case of particular interest is when $\epsilon(t)$ follows the Markov chain as proposed by Jarrow, Lando and Turnbull (1997) with default being an absorbing state. Here, the state space of this continuous time, time-homogenous

[12] As mentioned in the introduction, empiricial estimates of recovery rates are available in two forms: (1) as prices of the defaulted debt at the time of default, and (2) as realized recovery rates - prices of the defaulted debt at the emergence from financial distress.

Markov chain represents the possible credit classes, with $S-1$ being the highest and 1 being the lowest. The last state 0, represents default. The generator $(q_{ij})_{S\times S}$ is of the form

$$Q = (q_{ij})_{S\times S} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ q_{10} & -q_1 & q_{12} & \cdots & q_{1(S-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{(S-1)0} & q_{(S-1)1} & q_{(S-1)2} & \cdots & -q_{S-1} \end{pmatrix}$$

where $q_{ij} \geq 0$ for all i, j, and $q_i = \sum_{i\neq j} q_{ij}$. Since

$$P\{\epsilon(t+\Delta t) = j | \epsilon(t) = i\} = q_{ij}\Delta t + o(\Delta t), \text{ for } i \neq j,$$

the instantaneous arrival rate of default is given by $\lambda_t = q_{\epsilon(t)0}$. The simplest case is when $S = 2$ for which $\epsilon(t) = 1, 0$ correspond to "healthy" and "default," respectively.

In the following, we assume that the underlying price process follows the regime switching model of expression (1). The case of jump diffusion is essentially the same, except that the parameters θ_0 and σ_0 in the regime switching model will be replaced by θ and σ, respectively.

For investor $\mathcal{A}$, it is easy to compute the time t price $V^{\mathcal{A}}(t,T)$ for the risky bond with maturity T. Recall that x is the insolvency barrier, and consequently

Default by time t, i.e. $\tau \leq t$

$$V^{\mathcal{A}}(t,T) = \begin{cases} Re^{-\int_t^T r(s)ds} & X_t \leq x \\ e^{-\int_t^T r(s)ds}[(K-R)\psi(\theta_0, T-t, \frac{1}{\sigma_0}\log\frac{x}{X_t}) + R] & X_t > x. \end{cases}$$

No default prior to time t

$$\begin{aligned} V^{\mathcal{A}}(t,T) &= e^{-\int_t^T r(s)ds} E_t^{\mathcal{A}}[1\cdot 1_{\{\tau>T\}} + K\cdot 1_{\{\tilde{\tau}>T, \tau\leq T\}} + R\cdot 1_{\{\tilde{\tau}\leq T,\ \ \tau\leq T\}}] \\ &= e^{-\int_t^T r(s)ds}[E_t^{\mathcal{A}}[1_{\{\tau>T\}}] + RE_t^{\mathcal{A}}[1_{\{\tau\leq T\}}] \\ &\quad +(K-R)\int_t^T\int_x^\infty P(\tau\in ds, X_\tau\in dz|\mathcal{F}_t^{\mathcal{A}})\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})] \\ &= e^{-\int_t^T r(s)ds}[R + (1-R)P(\tau > T|\epsilon(t)\neq 0) \\ &\quad +(K-R)\int_t^T\int_x^\infty P(\tau\in ds, X_\tau\in dz|X_t, \epsilon(t)\neq 0)\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})]. \end{aligned}$$

In the above expression, $P(\tau > T|\epsilon(t) \neq 0)$ can be directly calculated following Jarrow, Lando Turnbull (1997). $P(\tau \in ds, X_\tau \in dz|X_t, \epsilon(t) \neq 0)$ may be computed explicitly in some cases, for example, it can be calculated via inverting the analytical expression of the Laplace transform for both the regime switching model (see Guo (2001b)) and the jump diffusion model with double exponential jumps (see Kou and Wang (2003)). When $S = 2$, the expression is further simplified to

$$\begin{aligned}
& V^{\mathcal{A}}(t,T) \\
= \; & e^{-\int_t^T r(s)ds}[R + (1-R)e^{-q_{10}(T-t)} \\
& +(K-R)\int_t^T \int_x^\infty dzdsq_{10}e^{-q_{10}(s-t)}\phi(\frac{\frac{1}{\sigma_1}\log\frac{z}{X_t} - \theta_1(s-t)}{\sqrt{s-t}}) \\
& \times\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})]
\end{aligned}$$

where ϕ is the density function of a standard normal random variable.

For investor $\mathcal{B}$, define $u = t_k \vee T_n < t < t_{k+1} \wedge T_{n+1}$, we have,

Default by time t Assuming default is an absorbing state, here $\tau \leq u < t$, and $\epsilon_u = \epsilon_t = 0$,[13] so

$$V^{\mathcal{B}}(t,T) = \begin{cases} Re^{-\int_t^T r(s)ds} & X_u \leq x \\ e^{-\int_t^T r(s)ds}[(K-R)\psi(\theta_0, T-u, \frac{1}{\sigma_0}\log\frac{x}{X_u}) + R] & X_u > x. \end{cases}$$

No default prior to time t Here $u < t$, but $\epsilon_u = \epsilon_t$, and

$$\begin{aligned}
V^{\mathcal{B}}(t,T) &= e^{-\int_t^T r(s)ds}E_t^{\mathcal{B}}[1\cdot 1_{\{\tau>T\}} + K\cdot 1_{\{\tilde{\tau}>T,\tau\leq T\}} + R\cdot 1_{\{\tilde{\tau}\leq T,\ \ \tau\leq T\}}] \\
&= e^{-\int_t^T r(s)ds}[E_t^{\mathcal{B}}[1_{\{\tau>T\}}] + RE_t^{\mathcal{B}}[1_{\{\tau\leq T\}}] \\
&\quad +(K-R)\int_t^T\int_x^\infty P(\tau\in ds, X_\tau\in dz|\mathcal{F}_t^{\mathcal{B}})\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})] \\
&= e^{-\int_t^T r(s)ds}[R + (1-R)P(\tau > T|\epsilon(t)\neq 0) \\
&\quad +(K-R)\int_t^T\int_x^\infty P(\tau\in ds, X_\tau\in dz|\mathcal{F}_t^{\mathcal{B}})\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})].
\end{aligned}$$

[13] Note that this assumes default is strictly before time t. On the default time $t = \tau$ we have that $V^{\mathcal{A}}(t,T) = V^{\mathcal{B}}(t,T)$. After the default time, however, these two values differ due to the delayed information for investor $\mathcal{B}$.

In particular,

$$\begin{aligned}
& V^{\mathcal{B}}(t,T) \\
= \; & e^{-\int_t^T r(s)ds}[R+(1-R)e^{-q_{10}(T-t)} \\
& +(K-R)\int_t^T \int_x^\infty dzdsq_{10}e^{-q_{10}(s-t)}\phi(\frac{\frac{1}{\sigma_1}\log\frac{z}{X_u}-\theta_1(s-u)}{\sqrt{s-u}}) \\
& \times\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})]
\end{aligned}$$

for $S=2$.

Here, the prices for $\mathcal{A}$ and $\mathcal{B}$ are different, and the latter is not a simple reparametrization of the former, although the underlying calculation uses the Markovian structure of $(X_t, \epsilon(t))$. This is because in case $\mathcal{B}$ the information concerning X_t is delayed at $u<t$, and the no-default condition gives an updated observation of $\epsilon(\cdot)$. Therefore, the exponential time between jumps for the Markov chain $\epsilon(t)$ and the independence between $\epsilon(t)$ and W_t are essential for the derivation.

4.3 A Comparison

This section compares risky debt prices between our extended model and the traditional approach. In our setting, once the firm defaults, the "recovery" rate is determined by the firm's asset value process. In the traditional model, the "recovery" rate is given exogenously as a known constant at the default time. These observations characterize the conceptual difference between these two approaches. To quantify this difference, let $V_C(t,T)$ denote the traditional value of the risky zero-coupon bond, which is

$$V_C^i(t,T) = e^{-\int_t^T r(s)ds}E_t^i[1_{\{\tau>T\}} + \widetilde{R}1_{\{\tau\le T\}}] \text{ for } t\le\tau$$

and $i\in\{\mathcal{A},\mathcal{B}\}$.

Let $V^i(t,T)$ denote their corresponding values under our recovery rate process. Recall from Proposition 2 that

$$V^i(t,T) = e^{-\int_t^T r(s)ds}E_t^i[1_{\{\tau>T\}} + R1_{\{\tau\le T\}} + (K-R)1_{\{\tau\le T,\tilde{\tau}>T\}}].$$

4.3.1 At Default, i.e. $\tau = t$

At the default date, the two model prices are:

$$V_C^i(\tau,T) = \widetilde{R}e^{-\int_\tau^T r(s)ds}$$

and

$$
\begin{aligned}
& V^{\mathcal{B}}(\tau,T) = V^{\mathcal{A}}(\tau,T) \\
= & \begin{cases} Re^{-\int_\tau^T r(s)ds} & \text{if insolvent} \\ e^{-\int_\tau^T r(s)ds}[(K-R)P(\inf_{\tau\le u\le T} X_u > x|\mathcal{F}_\tau^i) + R] & \text{if solvent} \end{cases}
\end{aligned}
$$

The debt's price in our model is a random variable at time τ if the firm is solvent, while in the traditional model it is a constant. Otherwise, the two model prices are equal. Hence, the two model prices will differ to the extent that the firm is solvent at that time of default.

For calibration purposes, readily available are the (average) market prices for defaulted debt at time of default τ, denoted M_τ, and at the time of emergence from financial distress, denoted M_∞. Estimates of the time duration that firms spend in financial distress (bankruptcy) can be found in Moody's (1999).[14]

Given the estimates $\{M_\tau, M_\infty\}$, a simple calibration procedure illustrates the differences between the parameters $\{\widetilde{R}, R, K\}$ in the traditional and extended models. In the traditional model, given estimates of default free rates, one can calibrate $\widetilde{R}$ by setting at time τ

$$M_\tau = \widetilde{R}e^{-\int_\tau^T r(s)ds}.$$

This is a relatively easy exercise. For our model, one would set

$$R = M_\infty$$

and implicitly estimate K by solving the following expression

$$M_\tau = e^{-\int_\tau^T r(s)ds}[(K - M_\infty)P(\inf_{\tau\le u\le T} X_u > x|\ \mathcal{F}_\tau^i) + M_\infty] \text{ if solvent}$$

i.e.

$$K = M_\infty + \frac{M_\tau e^{\int_\tau^T r(s)ds} - M_\infty}{P(\inf_{\tau\le u\le T} X_u > x|\ \mathcal{F}_\tau^i)}$$

or, in terms of $\widetilde{R}, R$:

$$K = R + \frac{\widetilde{R} - R}{P(\inf_{\tau\le u\le T} X_u > x|\ \mathcal{F}_\tau^i)}.$$

[14] The median time in bankruptcy for bankrupt firms over the time period 1982-1997 is 1.15 years with a standard deviation of 1.2 years.

This implicit estimation procedure will require a model for the asset price process. As mentioned earlier, direct estimates for K are not readily available. As evidenced by this discussion, the three calibrated parameters $\{K, \widetilde{R}, R\}$ will, in general, be unequal. Of course, other calibration schemes are possible, involving a more detailed usage of recovery rate data. These alternative procedures are left to subsequent research.

4.3.2 Prior to Default, i.e. $\tau > t$

Given the calibrated parameters $\{K, \widetilde{R}, R\}$, the difference between the extended and traditional model prices, prior to default is

$$\begin{aligned} V^i(t,T) - V_C^i(t,T) &= e^{-\int_t^T r(s)ds}(R - \widetilde{R})P(\tau \leq T|\mathcal{F}_t^i) \\ &\quad + e^{-\int_t^T r(s)ds}(K - R)P(\tilde{\tau} > T, \tau \leq T|\mathcal{F}_t^i) \end{aligned}$$

for $i \in \{\mathcal{A}, \mathcal{B}\}$. The difference between the two debt prices is that our extension reflects the recovery rate process over $[t, T]$, and it is determined by the evolution of the firm's asset value process. Indeed, in the traditional case, default implies that a fractional recovery $\widetilde{R}$ occurs with probability one; in contrast, our model allows the recovery in the event of default to be K or R, with a positive probability, depending on whether bankruptcy occurs before the debt's maturity.

Example: Jarrow-Lando-Turnbull. Now coming back to the Jarrow-Lando-Turnbull setting. Without loss of generality, we assume that X follows the regime switching model of expression (1) and $X_t = y$. For investor $\mathcal{A}$ we have the following expressions.

$$\begin{aligned} & V^{\mathcal{A}}(t,T) - V_C^{\mathcal{A}}(t,T) \\ = \; & e^{-\int_t^T r(s)ds}[(R - \widetilde{R})P(\tau \leq T|\epsilon(t) \neq 0) \\ & + (K - R)\int_t^T \int_x^\infty P(\tau \in ds, X_\tau \in dz|X_t, \epsilon(t) \neq 0)\psi(\theta_0, T - s, \frac{1}{\sigma_0}\log\frac{x}{z})]. \end{aligned}$$

If $S = 2$, then,

$$\begin{aligned} & V^{\mathcal{A}}(t,T) - V_C^{\mathcal{A}}(t,T) \\ = \; & e^{-\int_t^T r(s)ds}[(R - \widetilde{R})(1 - e^{q_{10}(T-t)}) \\ + \; & (K - R)\int_0^T \int_x^\infty e^{-q_{10}(s-t)} q_{10}\phi(\frac{\frac{1}{\sigma_1}\log\frac{z}{y} - \theta_1(s-t)}{\sqrt{s-t}})\psi(\theta_0, T - s, \frac{1}{\sigma_0}\log\frac{x}{z})dsdz]. \end{aligned}$$

The above expression shows that the difference in debt prices is an increasing function of y: the higher the asset value upon default, the bigger chance of recovery. it is also an increasing function of the drift μ_0, and a decreasing function of the volatility σ_0: the more volatile the asset value process, the less chance of recovery; it also depends on the maturity of the bond upon default, the closer t to T, the greater the chance of being solvent.

For investor $\mathcal{B}$, if there is no prior default at time t, then

$$\begin{aligned} & V^{\mathcal{B}}(t,T) - V_C^{\mathcal{B}}(t,T) \\ = \; & e^{-\int_t^T r(s)ds}[(R-\widetilde{R})P(\tau < T|\epsilon(t) \neq 0) \\ & +(K-R)\int_t^T \int_x^\infty P(\tau \in ds, X_\tau \in dz|\mathcal{F}_t^{\mathcal{B}})\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})]. \end{aligned}$$

The above formula is explicit for the case of $S = 2$, for which

$$\begin{aligned} & V^{\mathcal{B}}(t,T) - V_C^{\mathcal{B}}(t,T) \\ = \; & e^{-\int_t^T r(s)ds}[(R-\widetilde{R})(1-e^{-q_{10}(T-t)}) \\ + \; & (K-R)\int_0^T \int_x^\infty e^{-q_{10}(s-t)} q_{10}\phi(\frac{\frac{1}{\sigma_1}\log\frac{z}{y} - \theta_1(s-u)}{\sqrt{s-u}})\psi(\theta_0, T-s, \frac{1}{\sigma_0}\log\frac{x}{z})dsdz]. \end{aligned}$$

4.3.3 After Default, i.e. $\tau < t$

Here again, $\tau \leq u < t$. Under the Jarrow-Lando-Turnbull setting, it is easy to see

$$V^{\mathcal{A}}(t,T) - V_C^{\mathcal{A}}(t,T) = \begin{cases} (R-\widetilde{R})e^{-\int_t^T r(s)ds} & X_t \leq x \\ e^{-\int_t^T r(s)ds}[(K-R)\psi(\theta_0, T-t, \frac{1}{\sigma_0}\log\frac{x}{X_t}) + R - \widetilde{R}] & X_t > x. \end{cases}$$

$$V^{\mathcal{B}}(t,T) - V_C^{\mathcal{B}}(t,T) = \begin{cases} (R-\widetilde{R})e^{-\int_t^T r(s)ds} & X_u \leq x \\ e^{-\int_t^T r(s)ds}[(K-R)\psi(\theta_0, T-u, \frac{1}{\sigma_0}\log\frac{x}{X_u}) + R - \widetilde{R}] & X_u > x. \end{cases}$$

5 Conclusion

In this paper, we propose a simple continuous time model for bankruptcy and the recovery rate process, useful for pricing risky debt both before and after default. These processes are quantified under two information structures related to the firm's asset value: that held by the firm's management

(complete information), and that held by the market (partial information). Partial information is characterized by delayed knowledge of the firm's asset value. In our model, default is necessary to trigger the recovery rate process. The recovery rate process, if triggered, depends on the firm's asset value. If the debt matures before the firm becomes insolvent (defined as the firm's asset value falling below an insolvency barrier), then the debt is paid in full or at some fractional level. The fractional recovery when the firm is solvent exceeds the amount that would be paid if the firm becomes insolvent and enters bankruptcy.

Our model shows that when investors have complete information, and if the firm's asset value is below an insolvency threshold at the time of default, then the default and bankruptcy intensity are equal. However, if the asset value is above or equal to this critical level, then the default and bankruptcy intensities are distinct, and default does not necessarily lead to immediate bankruptcy. When investors have partial information, default and bankruptcy are both conceptually and analytically distinct. This implies that both risky debt and credit derivative valuation will differ in our model from that obtained using the traditional reduced form credit risk models. In contrast to the traditional reduced form models, our model is capable of pricing defaulted risk debt.

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6 Appendix A: Elementary property of one dimensional diffusion process via the scale measure

Without loss of the generality, we consider the simplest case when X is such that $X_t = \mu t + \sigma W_t$. Define

$$\tilde{\tau}_a^{\pm} = \inf\{t > 0, X_t = a \ \mid \ X_0 = a \pm \epsilon\}, \tag{7}$$

and let

$$q_s^+(\epsilon)ds = P_r(\tilde{\tau}_{a-\epsilon} > \tilde{\tau}_{a+\epsilon} \in ds), \qquad q_s^-(\epsilon)ds = P_r(\tilde{\tau}_{a+\epsilon} > \tilde{\tau}_{a-\epsilon} \in ds), \tag{8}$$

denote the probability density of the exit time of X_t from $(a-\epsilon, a+\epsilon)$, starting from $X_0 = a$.

Now, with ϕ bounded away from zero for $a-\epsilon \leq \xi \leq a+\epsilon$, the continuity of $s(x) = \exp[-2\mu \int_a^x (1/\phi(\xi))d\xi]$follows. Furthermore, $s(x)$ is bounded and bounded away from zero on $[a-\epsilon, a+\epsilon]$. A simple calculation via the scale measure shows that

$$\lim p_\epsilon = \lim \frac{\int_{a-\epsilon}^{a} s(x)dx}{\int_{a-\epsilon}^{a+\epsilon} s(x)dx} = \frac{1}{2},$$

(for example, see Karlin and Taylor (1981), Ch. 15, Equation (3.10)).

7 Appendix B: Accessible and Totally Inaccessible Parts of Stopping Times

In this section alone, with a bit abuse of notation, we will use τ for any general stopping time, including the default time and the bankruptcy time in the paper.

The most relevant result about the decomposition of a general stopping time τ is the following.

Theorem 5 *For every stopping time τ, there exists an $A \in \mathcal{F}_{\tau_-}$ such that $A \subset \{\tau < \infty\}$, and τ_A is accessible and τ_{A^c} is totally inaccessible. Such A is a.s. unique.*

The detailed proof can be found from He, Wang and Yan (1992). However, a few remarks for this Theorem are relevant here.

First, the key to the proof is to find the set A, from which the decomposition of τ into τ_A and τ_{A^c} is simple. Indeed, the two new stopping times τ_A and τ_{A^c} are stopping time τ restricted on the set A and A^c, and are generally referred to as the accessible part and totally accessible part of τ, respectively.

Secondly, the proof of the existence of A is constructive, as follows. Define

$$\mathcal{H} = \{\cup_n \{S_n = \tau < \infty\} : (S_n)_{(n\geq 1)} \text{ is a sequence of predictable times}\}.$$

Clearly $\mathcal{H} \subset \mathcal{F}_{\tau_-}$, $\mathcal{H}$ is closed under the formation of countable unions. It is not hard to see that there exists an $A \in \mathcal{H}$ such that $A = esssup\mathcal{H}$, for which τ_A is accessible and τ_{A^c} is totally inaccessible.

Lastly, note that $\tau_A = \tau I_A + (+\infty) I_{A^c}$, $\tau \leq \tau_A$ and $\tau \leq \tau_{A^c}$. Here $A^c = \Lambda$ for Λ used in the main text.

8 Appendix C: Proof of Theorem 2

Proof: First, for ease of exposition, denote by $\mathcal{F}_{k,n}$ the σ-field generated by $W_{t_1}, \cdots, W_{t_k}, T_1, \cdots, T_n, \epsilon(T_1), \cdots, \epsilon(T_n)$ and $W_{T_1}, \cdots, W_{T_n}$; and denote $P(\cdot|\mathcal{F}(y_k, u_n, x_n, v_n))$ for $P(\cdot|W_{t_1} = y_1, \cdots, W_{t_k} = y_k, T_1 = u_1, \cdots, T_n = u_n, W_{T_1} = x_1, \cdots, W_{T_n} = x_n, \epsilon(T_1) = v_1, \cdots, \epsilon(T_n) = v_n)$.

Note that on the event $\{\tau > t, T_n \leq t < T_{n+1}\}$, when $T_1, \cdots, T_n$, $\epsilon(T_1), \cdots,$ and $\epsilon(T_n)$ are known, there is a one-to-one correspondence between $(W_{t_1}, \cdots, W_{t_k}, W_{T_1}, \cdots, W_{T_n})$ and $(X_{t_1}, \cdots, X_{t_k}, X_{T_1}, \cdots, X_{T_n})$. We thus denote by x_i' the value of X_{T_i} $(1 \leq i \leq n)$ and by y_j' the value of X_{t_j} $(1 \leq j \leq k)$ given $\mathcal{F}(y_k, u_n, x_n, s_n)$. Finally, let $\theta_i = \frac{\mu_i}{\sigma_i} - \frac{\sigma_i}{2}$ $(0 \leq i \leq S-1)$, $X_t^{(i)} = \exp\{(\mu_i - \frac{\sigma_i^2}{2})t + \sigma_i W_t\}$ $(0 \leq i \leq S-1)$, and $(\mathcal{F}_t^W)_{t\geq 0}$ be the natural filtration of W.

First of all, by the Bayes' formula and the structure of the filtration $(\mathcal{F}_t)_{t\geq 0}$, we have, for $t \in [t_k, t_{k+1})$,

$$\begin{aligned} & P(t+h \geq \tilde{\tau} > t|\mathcal{F}_t) \\ = \; & 1_{\{\tilde{\tau}>t\}} \left(1 - \sum_{n\geq 0} 1_{\{T_{n+1}>t\geq T_n\}} \frac{P(\tilde{\tau} > t+h, T_{n+1} > t \geq T_n|\mathcal{F}_{k,n})}{P(\tilde{\tau} > t, T_{n+1} > t \geq T_n|\mathcal{F}_{k,n})} \right) \end{aligned}$$

Case (i): if $\epsilon(t) = 0$, then

$$
\begin{aligned}
& P(\tilde{\tau} > t, T_{n+1} > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= \; & P(x'_i \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{T_i}^{(0)}} > x, i \leq n-1, v_i = 0, x'_n \inf_{u_n \leq s \leq t} \frac{X_s^{(0)}}{X_{u_n}^{(0)}} > x | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
& \times P(T_{n+1} > t | T_n = u_n, \epsilon(T_n) = v_n) \\
= \; & \begin{cases} \text{if } u_n \geq t_k, \; P(x'_i \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\ \qquad e^{-q_0(t-u_n)} \psi(\theta_0, t-u_n, \frac{1}{\sigma_0} \log \frac{x}{x'_n}), \\ \text{if } u_n < t_k, \; P(x'_i \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0, \\ \qquad x'_n \inf_{u_n \leq s \leq t_k} \frac{X_s^{(0)}}{X_{u_n}^{(0)}} > x | \mathcal{F}(y_k, u_n, x_n, v_n)) \\ \qquad e^{-q_0(t-u_n)} \psi(\theta_0, t-t_k, \frac{1}{\sigma_0} \log \frac{x}{y'_k}). \end{cases}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \frac{P(\tilde{\tau} > t+h, T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}{P(\tilde{\tau} > t, T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})} \\
= \; & e^{-q_0 h} \frac{\psi(\theta_0, t+h-t_k \vee T_n, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})}{\psi(\theta_0, t-t_k \vee T_n, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})} + I + II,
\end{aligned}
$$

where

$$
I = \frac{P(\tilde{\tau} > t+h, T_{n+2} > t+h \geq T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}{P(\tilde{\tau} > t, T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})},
$$

$$
II = \frac{P(\tilde{\tau} > t+h, t+h \geq T_{n+2} > T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}{P(\tilde{\tau} > t, T_{n+1} > t \geq T_n | \mathcal{F}_{k,n})}.
$$

To calculate I, it is clear that

$$
\begin{aligned}
& P(\tilde{\tau} > t+h, T_{n+2} > t+h \geq T_{n+1} > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= & \sum_{j \neq 0} \int_t^{t+h} P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
& \times P(T_{n+2} > t+h | T_{n+1} = u_{n+1}, \epsilon(T_{n+1}) = j) \\
& \times P(T_{n+1} \in du_{n+1}, \epsilon(T_{n+1}) = j | T_n = u_n, \epsilon(T_n) = 0) \\
= & \begin{cases} \text{if } u_n \geq t_k,\ P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1,\ v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\ \qquad \sum_{j \neq 0} \int_t^{t+h} e^{-q_j(t+h-u_{n+1})} q_0 e^{-q_0(u_{n+1}-u_n)} \frac{q_{0j}}{q_0} \psi(\theta_0, u_{n+1} - u_n, \frac{1}{\sigma_0} \log \frac{x}{x_n'}) du_{n+1}, \\ \text{if } u_n < t_k,\ P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1,\ v_i = 0, \\ \qquad x_n' \inf_{u_n \leq s \leq t_k} \frac{X_s^{(0)}}{X_{u_n}^{(0)}} > x | \mathcal{F}(y_k, u_n, x_n, v_n)) \\ \qquad \sum_{j \neq 0} \int_t^{t+h} e^{-q_j(t+h-u_{n+1})} q_0 e^{-q_0(u_{n+1}-u_n)} \frac{q_{0j}}{q_0} \psi(\theta_0, u_{n+1} - t_k, \frac{1}{\sigma_0} \log \frac{x}{y_k'}) du_{n+1}. \end{cases}
\end{aligned}
$$

So I is

$$
\frac{\sum_{j \neq 0} \int_t^{t+h} e^{-q_j(t+h-u_{n+1})} q_{0j} e^{-q_0(u_{n+1}-T_n)} \psi(\theta_0, u_{n+1} - T_n \vee t_k, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}}) du_{n+1}}{e^{-q_0(t-T_n)} \psi(\theta_0, t - T_n \vee t_k, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})},
$$

and $\frac{I}{h} \to \sum_{j \neq 0} q_{0j} = q_0$ as $h \downarrow 0$.

Similarly, $II \leq Ch^2$ for some constant C, and

$$
\lim_{h \downarrow 0} \frac{P(t+h \geq \tilde{\tau} > t | \mathcal{F}_t)}{h} = -\frac{\psi_t(\theta_0, t - t_k \vee T_n, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})}{\psi(\theta_0, t - t_k \vee T_n, \frac{1}{\sigma_0} \log \frac{x}{X_{t_k \vee T_n}})}.
$$

Case (ii): if $\epsilon(t) \neq 0$, then

$$
\begin{aligned}
& P(\tilde{\tau} > t, T_{n+1} > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= & P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) e^{-q_{v_n}(t-u_n)},
\end{aligned}
$$

$$
\begin{aligned}
& P(\tilde{\tau} > t+h, T_{n+1} > t+h > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= \; & P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) e^{-q_{v_n}(t+h-u_n)},
\end{aligned}
$$

and

$$
\begin{aligned}
& P(\tilde{\tau} > t+h, T_{n+2} > t+h \geq T_{n+1} > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= \; & P(\tilde{\tau} > t+h, T_{n+2} > t+h \geq T_{n+1} > t \geq T_n, \epsilon(T_{n+1}) \neq 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
& + P(\tilde{\tau} > t+h, T_{n+2} > t+h \geq T_{n+1} > t \geq T_n, \epsilon(T_{n+1}) = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
= \; & I + II,
\end{aligned}
$$

with

$$
\begin{aligned}
I \;=\; & P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
& \times \sum_{j \neq 0, \epsilon(T_n)} \int_t^{t+h} q_{\epsilon(T_n) j} e^{-q_{\epsilon(T_n)}(u - T_n)} e^{-q_j(t+h-u)} du,
\end{aligned}
$$

and

$$
\begin{aligned}
II \;=\; & \int_t^{t+h} P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0, \epsilon(T_{n+1}) = 0, \\
& x_n' e^{(\mu_{v_n} - \frac{\sigma_{v_n}^2}{2})(u_{n+1} - u_n) + \sigma_{v_n}(W_{u_{n+1}} - W_{u_n})} \inf_{u_{n+1} \leq s \leq t+h} \frac{X_s^{(0)}}{X_{u_{n+1}}^{(0)}} > x | \mathcal{F}(y_k, u_n, x_n, v_n), T_{n+1} = u_{n+1}) \\
& \times P(T_{n+2} > t+h | T_{n+1} = u_{n+1}) \times P(T_{n+1} \in du_{n+1} | T_n = u_n, \epsilon(T_n) = v_n) \\
= \; & \begin{cases}
\text{if } u_n \geq t_k, \; P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
\qquad \frac{q_{v_n 0}}{q_{v_n}} \int_t^{t+h} e^{-q_0(t+h-u_{n+1})} q_{v_n} e^{-q_{v_n}(u_{n+1} - u_n)} \\
\qquad E[\psi(\theta_0, t+h-u_{n+1}, \frac{1}{\sigma_0} \log \frac{x}{x_n'} - \frac{\sigma_{v_n}}{\sigma_0} W_{u_{n+1}-u_n}^{(\theta_{v_n})})] du_{n+1} \\
\text{if } u_n < t_k, \; P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
\qquad \int_t^{t+h} e^{-q_0(t+h-u_{n+1})} q_{v_n} e^{-q_{v_n}(u_{n+1} - u_n)} \frac{q_{v_n 0}}{q_{v_n}} \\
\qquad E[\psi(\theta_0, t+h-u_{n+1}, \frac{1}{\sigma_0} \log \frac{x}{y_k'} - \frac{\sigma_{v_n}}{\sigma_0} W_{u_{n+1}-t_k}^{(\theta_{v_n})})] du_{n+1}.
\end{cases}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
& P(\tilde{\tau} > t+h, t+h \geq T_{n+2} > T_{n+1} > t \geq T_n | \mathcal{F}(y_k, u_n, x_n, v_n)) \\
\leq \quad & P(x_i' \inf_{u_i \leq s \leq u_{i+1}} \frac{X_s^{(0)}}{X_{u_i}^{(0)}} > x, i \leq n-1, v_i = 0 | \mathcal{F}(y_k, u_n, x_n, v_n)) e^{-q_{v_n}(t-u_n)} C h^2,
\end{aligned}
$$

for some constant C. Therefore,

$$
\begin{aligned}
& \lim_{h \downarrow 0} \frac{P(t+h \geq \tilde{\tau} > t | \mathcal{F}_t)}{h} \\
= \quad & q_{\epsilon(t)0} \Phi \left(\frac{\frac{1}{\sigma_{\epsilon(t)}} \log \frac{x}{X_{T_n \vee t_k}} - \theta_{\epsilon(t)}(t - T_n \vee t_k)}{\sqrt{t - T_n \vee t_k}} \right).
\end{aligned}
$$

LABELS	EVENTS	RECOVERY RATES
no default	$\epsilon(t) \neq 0$	NA
default	$\epsilon(t) = 0$	NA
solvent	$X_t \geq x$	NA
insolvent	$X_t < x$	NA
not bankrupt	$\{\epsilon(t) \neq 0\} \cap \{X_t \geq x\}$	1
	$\{\epsilon(t) \neq 0\} \cap \{X_t < x\}$	1
	$\{\epsilon(t) = 0\} \cap \{X_t \geq x\}$	K
bankrupt	$\{\epsilon(t) = 0\} \cap \{X_t < x\}$	R

Table 1: The Events and Recovery Rates of the Firm at time t.

Bankruptcy occurs when the firm is in default and the firm becomes insolvent. Either default or insolvency alone does not induce bankruptcy. The parameters satsify $R \leq K \leq 1$.