On valuation before and after tax in no arbitrage models: Tax neutrality in the continuous time model

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Abstract: We establish necessary and sufficient conditions for a linear taxation system to be neutral − within the continuous-time "no arbitrage" model − in the sense that asset valuation is invariant to the process for tax rates and choice of realization dates as well as immune to timing options attempting to twist the time profile of taxable income through wash sale transactions. We also demonstrate that despite neutrality the portfolio choice can be quite different across investors subject to different tax rates.

1 Introduction

The finance literature on asset pricing, theoretical as well as empirical, is surprisingly void of tax considerations. Tax considerations play a very important role in most real world investment decisions, whether financial or real, and tax issues are a cornerstone in corporate finance and capital structure theory. However, asset pricing theory has little to say about the effects of different tax systems on asset demand functions and market equilibrium prices. Widely used textbooks on asset pricing theory do not even have the word ≫taxation≪ or related topics in their index, and a similar situation is found in the fixed-income literature.

There may be a variety of explanations for this. One explanation could be that taxation issues are quite complicated to handle in any theoretical model.1) It is often hard to obtain nice theoretical results when aspects of taxation are included.2) Taxation induces a certain individual element into return distributions and pricing relations, because return distributions and discount factors depend on the tax rules. If arbitrage opportunities after tax exist, some constraints like ≫limits to tax deductability≪, ≫limits to short positions≪ or other asset allocation restrictions are necessary in order to limit the extent to which such arbitrage opportunities can be exploited. The resulting market situation will be one marked by corner solutions and clientele effects which are inherently complicated to model. While this is not a good explanation for avoiding taxation in asset pricing models there may well be some truth in it.

Another explanation could be that taxation is considered as irrelevant for asset pricing because taxation is neutral in the sense that pricing relations on an after tax basis in accordance with a taxable investor's preferences lead to the same asset prices and investment decisions as in a world without taxes. If this is true it is clearly of interest to know what such a neutral tax system looks like − a discussion that we have not found to any extent in the finance literature. This paper provides such a discussion for the standard continuous time model.

Taxation rules which lead to agreement about asset prices across investors subject to different tax rates and simultaneously eliminate the profitability of portfolio dispositions solely made in order to avoid or defer tax payments are known as neutral systems in the field of public economics. A neutral taxation system is considered as a normative benchmark with which other taxation systems may be compared and the severeness of deviations measured. However, the public economics literature has only vaguely made use of the analytical techniques developed in mathematical finance in order to characterize "absence of arbitrage" in financial markets. This paper shows that these techniques can be quite powerful outside a narrowly defined territory of finance theory.

Clearly, the whole structure of prices and return distributions could − in a more fundamental sense − change as a result of introducing or changing taxes. Obvious channels for such consequences are wealth effects or wealth redistribution effects; additionally, taxation involves a risk sharing mechanism between the government and the taxpaying investor which may be affected when taxes are changed. However, such issues are not at stake here. Market prices are market prices for taxable investors as well as for tax free investors, and the point of departure is that market prices as well as certain types of investors with different tax rates exist. We want to examine − in the continuous time framework − whether such market prices can be consistent with heterogeneity across investors with respect to taxation in the sense that no arbitrage opportunities exist.

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1) This is the case in the real world as well!

2) As noted by Dybvig and Ross (1986) in their introduction: ≫In the study of investments, taxes are largely a source of embarrassment to financial economists.≪

We ask and answer the following simple question: If a set of asset prices and return distributions presents a "no arbitrage" equilibrium before tax, under what conditions will it also be a "no arbitrage" equilibrium after tax for an investor subject to a linear and symmetric tax schedule. Or, more generally, if a set of asset prices and return distributions presents a "no arbitrage" equilibrium after tax for *some* taxable investor subject to a linear and symmetric tax schedule, under what conditions will it also be a "no arbitrage" equilibrium after tax for any other taxable investor subject to a linear and symmetric tax schedule?

We use the standard Black-Scholes type of model as the foundation for this analysis. The focus in this paper is on the essentials of taxation neutrality and not on how general the assumptions for the price processes can be for the results to remain true. The first taxation system to be analyzed is the mark-to-market taxation system, which is used as a vehicle for our more general discussion. We then continue to provide necessary and sufficient conditions for a given taxation system to be neutral and show how some of the taxation systems suggested in the public economics literature fit into our framework. We are able to specify exactly the degree of uncertainty allowed for in the interest rate process as well as in the process describing the development of the tax rates over time − an issue largely neglected in the public economics literature. Additionally, different taxation systems may lead to neutrality in the pricing of financial assets, despite the fact that the induced risk sharing mechanism between the investor and the government may be quite different and lead to differences in investor behaviour towards market risk.

Dealing with taxation in the framework of a dynamic model raises the question about how to account for accrued capital income as taxable income. Most of the finance literature has dealt with a discrete time one-period framework, where investment takes place at the beginning of the period and capital income is earned and taxed at the end of the period. The seminal contributions by Schaefer, cf. Schaefer (1981, 1982a,b), belong to this class of models. Other examples are Dybvig and Ross (1986), Dammon (1987), Dammon and Green (1987), Ross (1987), Dermody and Prisman (1988) and Dermody and Rockafeller (1991). One-period models do not give rise to any problems with accruals. Capital income is taxed upon realization and accrued capital income does not exist in a one-period model. The conclusions from the one-period framework, however, cannot be extrapolated to a dynamic framework without careful considerations.

A few papers deal with multi-period problems, but then capital income is taxed whenever it is realized − i.e. at the point in time where it has cash flow consequences. Seminal contributions in this direction encompass Constantinides (1983) and Constantinides and Ingersoll (1984a,b). Other examples are Dermody and Rockafeller (1995), Dammon and Spatt (1996), and Cadenillas and Pliska (1999). For such multi-period models it is well-known that the tax rules give rise to lock-in effects and that tax arbitrage problems are difficult to avoid unless some restrictions − short selling constraints, e.g. − are imposed. Even so, taxable investors may be left with timing options. I.e., for any given series of market price movements a taxable investor can influence to her own advantage the timing of gains and losses as taxable income. A *wash sale* where assets are sold and immediately repurchased with the sole purpose of generating a tax deferral is a well-known example of such timing options.

The paper is organized as follows. The basic results from the mark-to-market taxation principle are analyzed in section 2. Section 3 gives a number of candidates that are neutral taxation systems with somewhat different properties. In section 4 our main theorem 1 is stated. Section 5 describes the implications for portfolio selection within the classical Merton (1971) model arising from different neutral taxation systems. Section 6 summarizes the paper. Parts of the proofs involving lengthy mathematical derivations are found in the Appendix.

2 The Black Scholes model and mark-to-market taxation

The premier example of a linear and neutral taxation system is to tax investors in accordance with mark-to-market valuation of their portfolio holdings. This section describes the workings of this taxation principle within the Black-Scholes model. In order to understand the consequences of taxation in a continuous time model this is a fully adequate illustration, at least to the extent that the underlying price-processes have continuous realizations. We demonstrate rigorously − within the "no arbitrage paradigm" − that that this taxation principle is in fact neutral. In section 3 we will characterize mark-to-market taxation as one special example within the family of neutral taxation systems.

As for this section we assume that the tax rate is a constant denoted by T. Similarly, we assume that the rate of interest is a constant r. Under the mark-to-market taxation principle the relevant rate of interest after tax for the bank account is $r(1-T)$ to be denoted by $r^{a.t.}$.

Taxes can be paid by liquidating part of the asset holding. Symmetrically, capital losses induce tax refunds that can be reinvested in the portfolio. Another way to implement a taxation system that avoids the need for an excessive amount of transactions due to tax bills is to account for tax liabilities in an earmarked and interest bearing account that may be settled whenever the portfolio is realized. Whether taxes are paid continuously or accumulate in an earmarked and interest bearing account is a matter of indifference with respect to valuation. Investors can always pay their tax bill by transacting in the bank account instead of transacting in the underlying portfolio if they desire to do so. We denote the time t value of such a tax account as A_t^S − the superscript indicates that the net tax payments are specific to a given asset or a given portfolio strategy.

With these assumptions the basics of this tax system is given in the equations (2.1) -(2.3) below.

$$\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dZ_t; \quad S_0 \; given && (2.1)\\
dA_t^S &= r^{a.t.} A_t^S dt + T dS_t \\
&= r^{a.t.} A_t^S dt + \mu T S_t dt + \sigma T S_t dZ_t; \quad A_0^S = 0 && (2.2)\\
d(S_t - A_t^S) &= \mu(1-T)[S_t - A_t^S]dt + [\mu(1-T) - r^{a.t.}]A_t^S dt \\
&\quad + \sigma(1-T)[S_t - A_t^S]dZ_t + \sigma(1-T)A_t^S dZ_t && (2.3)
\end{aligned}$$

The basic lognormal price process for the risky asset is given in (2.1). The development in the tax account is given in (2.2); any outstanding balance (positive as well as negative) is charged with a taxable/tax-deductible interest and the accrued capital gain dS_t is taxed by the tax rate T.

These relations form a simultaneous system of two stochastic differential equations. Writing them together, introducing the risk-neutralized Brownian motion $\widetilde{Z}_t$, leads to:

$$\begin{aligned}
d\begin{bmatrix} S_t - A_t^S \\ A_t^S \end{bmatrix} &= \begin{bmatrix} \mu(1-T) & (\mu - r)(1-T) \\ \mu T & \mu T + r^{a.t.} \end{bmatrix} \begin{bmatrix} S_t - A_t^S \\ A_t^S \end{bmatrix} dt \\
&\quad + \begin{bmatrix} \sigma(1-T) & \sigma(1-T) \\ \sigma T & \sigma T \end{bmatrix} \begin{bmatrix} S_t - A_t^S \\ A_t^S \end{bmatrix} dZ_t \\
&= \begin{bmatrix} r^{a.t.} & 0 \\ rT & r \end{bmatrix} \begin{bmatrix} S_t - A_t^S \\ A_t^S \end{bmatrix} dt + \\
&\quad + \begin{bmatrix} \sigma(1-T) & \sigma(1-T) \\ \sigma T & \sigma T \end{bmatrix} \begin{bmatrix} S_t - A_t^S \\ A_t^S \end{bmatrix} d\widetilde{Z}_t && (2.4)
\end{aligned}$$

where

$$\widetilde{Z}_t \equiv B_t + \frac{\mu - r}{\sigma} t \tag{2.5}$$

is a Brownian motion under the usual risk-neutral measure. The solution to (2.4) is

$$\begin{aligned}
\begin{bmatrix} S_t - A_t^S \\ A_t^S \end{bmatrix} &= \begin{bmatrix} e^{tr^{a.t.}} \\ e^{rt} - e^{tr^{a.t.}} \end{bmatrix} S_0 + \\
&\int_0^t \begin{bmatrix} (1-T)e^{(t-u)r^{a.t.}} \\ e^{r(t-u)} - (1-T)e^{(t-u)r^{a.t.}} \end{bmatrix} \sigma S_u d\widetilde{Z}_u && (2.6) \\
e^{-tr^{a.t.}}(S_t - A_t^S) &= S_0 + \int_0^t \sigma(1-T)e^{-ur^{a.t.}} S_u d\widetilde{Z}_u && (2.7)
\end{aligned}$$

From equations (2.6)-(2.7) we observe that the risk-neutralized probability measure valid on a before tax basis, i.e. the measure under which $\widetilde{Z}_t$ is a Brownian motion, is also valid on an after-tax basis in the following sense: Before tax the discounted price process $e^{-rt}S_t$ is a martingale under this measure if and only if the relevant discounted after tax value process $e^{-r(1-T)t}(S_t - A_t^S)$ is a martingale for any given tax rate T.

The usual PDE derived from a perfect hedge argument can also be obtained. Consider any contingent claim with market price $C(S_t, t)$ and associated tax account A_t^C. The relevant dynamics and the standard hedge argument goes as follows:

$$\begin{aligned}
dC(S_t,t) &= [C_t^{'}dt + C_S^{'}\mu S_t + \frac{1}{2}C_{SS}^{''}\sigma^2 S^2]dt + C_S^{'}\sigma S_t dZ_t \\
&\equiv \mathcal{L}(C)dt + C_S^{'}\sigma S_t dz_t \\
dA_t^C &= r^{a.t.}A_t^C dt + TdC(S_t,t) \\
&= r^{a.t.}A_t^C dt + T\mathcal{L}(C)dt + TC_S^{'}\sigma S_t dZ_t; \quad A_0^C = 0 \\
d(C(S_t,t) - A_t^C) &= (1-T)\mathcal{L}(C)dt + (1-T)C_S^{'}\sigma S_t dZ_t - r^{a.t.}A_t^C dt \\
d(C(S_t,t) - A_t^C) - C_S^{'}d\left(S_t - A_t^S\right) &= (1-T)\mathcal{L}(C)dt + (1-T)C_S^{'}\sigma S_t dZ_t - r^{a.t.}A_t^C dt \\
&\quad -C_S^{'}\left[\mu(1-T)S_t - r^{a.t.}A_t^S dt + \sigma_S(1-T)S_t dZ_t\right] \\
&= (1-T)\left[C_t^{'} + \frac{1}{2}C_{SS}^{''}\sigma^S S_t^2 - r\left(A_t^C - C_S^{'}A_t^S\right)\right]dt
\end{aligned} \tag{2.8}$$

By the standard "no arbitrage" argument the return on the risk-free position created by the hedge must equal the risk-free return on an after tax basis. I.e.

$$(1-T)\left[C_t^{'} + \frac{1}{2}C_{SS}^{''}\sigma^S S_t^2 - r\left(A_t^C - C_S^{'}A_t^S\right)\right]dt = r^{a.t.}\left[C(S_t,t) - A_t^C - C_S^{'}\left(S_t - A_t^S\right)\right]dt \tag{2.9}$$

Upon reduction all tax accounts clear and the usual PDE before tax appears:

$$\frac{1}{2}C_{SS}^{''}\sigma^S S_t^2 + r\left[C_S^{'}S_t - C\right] + C_t^{'} = 0 \tag{2.10}$$

I.e., the PDE derived on an after tax basis from applying standard hedging arguments is identical to the usual PDE on a before tax basis. Hence, this form of taxation is consistent with a partial equilibrium pricing theory that ignores taxation.

If taxes were to be paid through transactions in the underlying assets alone the equivalent dynamics would otherwise have to separate the development of market prices from the development of the investor's net wealth. We denote the latter by $V(S_t, t)$ for an asset bought at time 0 at the market price S_0. The process for $V(S_t, t)$ is then

$$dV(S_t, t) = \mu(1-T)V(S_t, t)dt + \sigma(1-T)V(S_t, t)dZ_t, \quad V(S_0) \equiv S_0 \tag{2.11}$$

with solution

$$V(S_H, H) = S_0 \exp\left\{\left(\mu(1-T) - \frac{1}{2}\sigma^2(1-T)^2\right)H + \sigma(1-T)Z_H\right\} \Leftrightarrow \tag{2.12}$$

$$V(S_H, H) = S_0 \exp\left\{\left(r^{a.t.} - \frac{1}{2}\sigma^2(1-T)^2\right)H + \sigma(1-T)\left(Z_H + \frac{\mu - r}{\sigma}H\right)\right\} \Leftrightarrow \tag{2.13}$$

$$V(S_H, H) = S_0 \exp\left\{\left(r^{a.t.} - \frac{1}{2}\sigma^2(1-T)^2\right)H + \sigma(1-T)\widetilde{Z}_H\right\} \tag{2.14}$$

Taking mean values w.r.t. to the equivalent martingale measure Q in (2.14) leads to the following two relations:

$$S_0 = \exp^{-r^{a.t.} \cdot H} E^Q\left[V(S_H, H)\right] \tag{2.15}$$

The neutrality result appears again. The discount factor has changed, but the usual change of measure rendering the process $\widetilde{Z}_t$ a Brownian motion remains unchanged by the introduction of taxation.

The above derivation were not in any fundamental way depending on the specifics of the geometric Brownian motion and the Black-Scholes model; the assumption of constant parameters is easily abandoned and the same is true for the assumed constant tax rate T. Any adapted process T_t that does not conflict with the necessary regularity requirements will lead to the same conclusion: If prices are in accordance with a given equivalent martingale measure for any investor with a given tax status, subject to a linear and symmetric tax process T_t, they will also be in accordance with the same equivalent martingale measure for any other investor with any other linear and symmetric tax process $\widehat{T}_t$.

Mark-to-market taxation is one among many possible neutral taxation systems. The following section 3 describes a variety of such systems some of which have real life counterparts in actual tax legislation. Necessary and sufficient conditions for a linear and symmetric taxation system to be neutral are derived in section 4.

3 Examples of linear taxation systems in continuous time

A linear tax system in continuous time is defined here as a tax system with the property that the taxes levied on the investment during any given holding period $[0, t]$ within an assumed time horizon $[0, H]$ is a linear functional on the observed values of the asset in $[0, t]$. Such a taxation system is obviously also linear in portfolio formation; and since derivatives can be reproduced by self-financing portfolio strategies in the risky assets and the bank account we only state the results for a single risky asset with price process S_t given by

$$dS_t = \mu(S_t, t)S_t dt + \sigma(S_t, t)S_t dZ_t \equiv r_t S_t dt + \sigma(S_t, t)S_t(dZ_t + \lambda_t dt) \tag{3.1}$$

where the risk premium λ_t is defined in the usual manner:

$$\lambda_t \equiv \frac{\mu(S_t, t) - r_t S_t}{\sigma(S_t, t)} \tag{3.2}$$

Everything is assumed to take place within a standard probability theoretic setup with the usual regularity conditions and with the filtration $\{\mathcal{F}_s\}_{s=0}^{s=H}$ taken as the filtration generated by the price processes for the risky assets and the bank account.

A family of linear tax functionals is given by a family of (signed) measures, $K_{tu},\ 0 \leq u \leq t$. These measures may be deterministic or – in case they are stochastic – the processes $u \rightarrow K_{tu}$ are required to be $\mathcal{F}_t$-adapted. Alternatively stated, the tax functionals are $\mathcal{F}_t$-adapted bounded variation processes. The economic interpretation of this requirement is clear. The tax consequences of a liquidation decision at time t must be known at time t. We will see later how this implies that K_{tu} must actually fulfill the somewhat stronger requirement of being $\mathcal{F}_u$-adapted.

The general mechanics of this can be rewritten in terms of Fubini's theorem for stochastic integrals, cf. Ikeda and Watanabe (1989), lemma 4.1. We state this for an investment in a single risky asset and continue to denote the tax account, where all intermediate taxes are discounted forward, by A_t^S; if the investor chooses to realize the position at time t her net result is $S_t - A_t^S$. A linear taxation system is one where this net result after tax is a linear functional of observed values during $[0, t]$:

$$S_t - A_t^S = \int_0^t S_u dK_{tu} = S_t K_{tt} - S_0 K_{t0} - \int_0^t K_{tu} dS_u \tag{3.3}$$

Alternatively, such a tax system can be characterized by the relation

$$A_t^S = S_t - \int_0^t dK_{tu} S_u = (1 - K_{tt}) S_t + S_0 K_{t0} + \int_0^t K_{tu} dS_u \tag{3.4}$$

which is also a linear functional of observed values during $[0, t]$.

One interpretation of (3.4) is that the terms $1 - K_{tt}$ and K_{t0} relate to tax consequences at buying and selling dates whereas $K_{tu},\ 0 \leq u < t$ relate to tax consequences arising from periodic capital gains and losses. We will return to this interpretation in the following section 4 and, in particular, in theorem 2.

Below we list some examples of linear tax systems known in the literature. Although not stated explicitly, the infinitesimally riskless asset may well have a stochastically varying rate of return r_t, which is taxed according to a tax rate T_t. r_t as well as T_t are required to be $\mathcal{F}_t$-adapted processes.

We work with the bank account before tax B_t as well as the bank account after tax $B_t^{a.t.}$ with the dynamics

$$dB_t = r_t B_t dt, \qquad dB_t^{a.t.} = r_t (1 - T_t) B_t^{a.t.} dt \tag{3.5}$$

Below we give some examples of what such linear taxation systems might look like. We also state the wealth dynamics after tax for an investor who is building up a liability of unsettled tax payments. This demonstrates the risk sharing mechanism between the investor and the government. These example will appear as special cases of taxation systems satisfying our general necessary and sufficient conditions in theorem 1.

Examples

1. **“Government takes all risk”**

$$\begin{aligned} S_t - A_t^S &= e^{\int_0^t r_u(1-T_u)du} S_0 = B_t^{a.t.} S_0 & (3.6) \\ K_{tu} &= 1_{\{u>0\}} B_t^{a.t.} & (3.7) \\ d(S_t - A_t^S) &= r_t(1-T_t) e^{\int_0^t r_u(1-T_u)du} S_0 dt = r_t(1-T_t)(S_t - A_t^S)dt & (3.8) \end{aligned}$$

This taxation system allows the investor to earn exactly the same riskless return after tax as would have been earned by investing the same initial amount of money in the riskless asset.

The family of measures given by K_{tu} induce a step function for each value of t. There is a single jump at $u = 0$ given by $B_t^{a.t.}$, because the initial value S_0 of the asset is the only property of the price process that matters. The rest is given by the development of the bank account after tax.

2. **Imputed wealth tax**

The imputed wealth tax charges a tax of the magnitude $r_u T_u S_u du$ at time u. If the tax payment is left in the tax account A_t^S it is carried forward with the discount factor after tax:

$$\begin{aligned} A_t^S &= \int_0^t r_u T_u \frac{B_t^{a.t.}}{B_u^{a.t.}} S_u du & (3.9) \\ S_t - A_t^S &= S_t - \int_0^t r_u T_u \frac{B_t^{a.t.}}{B_u^{a.t.}} S_u du & (3.10) \\ K_{tu} &= \frac{T_u}{1-T_u} \frac{B_t^{a.t.}}{B_u^{a.t.}} + 1_{\{u=t\}} & (3.11) \end{aligned}$$

Each of the measures given by K_{tu} are absolutely continuous wrt time on the interval $[0, t)$. The jump at the diagonal $u = t$ is in order to neutralize any separate taxation effects from realization, which is contained in $1 - K_{tt}$.

The wealth dynamics shows that the investor receives the riskless rate of return on her *net wealth* $S_t - A_t^S$ and the risk as well the risk premium on her *gross wealth* (her before tax position) S_t:

$$\begin{aligned} d(S_t - A_t^S) &= (\mu - r_t T_t) S_t dt + \sigma_t S_t dZ_t - r_t(1-T_t) A_t dt \\ &= (\mu - r_t) S_t dt + \sigma_t S_t dZ_t + r_t(1-T_t)(S_t - A_t^S)dt \\ &= r_t(1-T_t)(S_t - A_t^S)dt + \sigma_t S_t (dZ_t + \lambda_t dt) & (3.12) \end{aligned}$$

3. **Auerbach (1991)** (**“the retrospective capital gains taxation system”**)

The proposal given in Auerbach (1991) for a retrospective capital gains tax that was paid only at realization, but maintained neutrality by avoiding gains from deferring the realization, is characterized by the atomic measure in (3.14):

$$\begin{aligned} A_t^S &= (1 - \frac{B_t^{a.t.}}{B_t}) S_t & (3.13) \\ K_{tu} &= 1_{\{u=t\}} \frac{B_t^{a.t.}}{B_t} & (3.14) \end{aligned}$$

The original idea in Auerbach's system was to impose a pure cash flow taxation. Taxes are only paid at points in time where the investor triggers a realization; whether this is due to a need for drawing down wealth for consumption purposes or an attempt to carry out a wash sale is irrelevant. This is reflected in the fact that each of the measures given by K_{tu} have a single jump at the diagonal $u=t$ given by $B_t^{a.t.}/B_t$ and no mass anywhere else.

As will become clear from theorem 1 in the following section it is also the only way one is able to construct a neutral taxation system with this requirement.

Looking at the dynamics of the "net position" $S_t - A_t^S$ this can be interpreted as a price process where dividends are taken away at the rate $r_t T_t$ continuously from the net position:

$$\begin{aligned}
A_t^S &= \left(1 - \frac{B_t^{a.t.}}{B_t}\right) S_t && (3.15)\\
S_t - A_t^S &= \frac{B_t^{a.t.}}{B_t} S_t && (3.16)\\
d(S_t - A_t^S) &= (S_t - A_t^S)\,((\mu - r_t T)dt + \sigma_t dZ_t) && (3.17)\\
d(S_t - A_t^S) &= (S_t - A_t^S)\,[r_t(1 - T_t)dt + \sigma_t(dZ_t + \lambda_t dt)] && (3.18)
\end{aligned}$$

This is similar to the imputed wealth taxation. However, the risk sharing mechanism between the investor and the government is different. The investor only carries the risk associated with her net wealth.

4. **Auerbach and Bradford (2001)** (and **Bradford (1995)**)

   This is a convex combination of the former two. I.e.

$$\begin{aligned}
K_{tu} &= H1_{\{u>0\}} B_t^{a.t.} + (1 - H)1_{\{u=t\}} \frac{B_t^{a.t.}}{B_t} && (3.19)\\
d(S_t - A_t^S) &= r_t(1 - T_t) H S_0 B_t^{a.t.} + (1 - H)\frac{B_t^{a.t.}}{B_t} S_t\,[(\mu - r_t T_t)dt + \sigma_t dZ_t] && (3.20)\\
&= r_t(1 - T_t)(S_t - A_t^S)dt + (1 - H)\frac{B_t^{a.t.}}{B_t} S_t \sigma_t\,[dZ_t + \lambda_t dt] && (3.21)
\end{aligned}$$

   The wealth dynamics shows that the investor receives the riskless rate of return on her *net wealth* $S_t - A_t^S$. In addition he shares the risk and the risk premium with the government in accordance with the weight attached to the Auerbach (1991) part of the net wealth position.

5. **Realization principle**

   The standard realization principle is **not** a neutral system. This will become clear when checked against the necessary and sufficient criteria in theorem 1. Assuming that the same tax rate T applies to the initial investment outlay and the realization value it can be described in terms of the notation used here as

$$K_{tu} = T1_{\{u>0\}} + (1 - T)1_{\{u=t\}} \qquad (3.22)$$

   Compared with the Auerbach and Bradford (2001) system it arises as a special case with $H=T$ under two assumptions: (i) the risk-free rate of interest is zero and (ii) the tax rate T_t is a constant. However, when the risk-free rate of interest is not zero there is a well known and valuable timing option on behalf of the investor. When the tax rate is allowed to vary stochastically the value of this timing option is less obvious.

A variant of the realization principle is the pure cash flow tax due to Brown (1948)[3)]. In Brown's system interest payments are neither taxable nor deductible, i.e. $B_t^{a.t.} = B_t$, and the basis for taxing any capital gain is not the original acquisition price S_0 but rather the original acquisition price increased by an imputed interest allowance, i.e. $S_0 B_t$. This is in accordance with the Auerbach and Bradford (2001) with $H = T$ and $B_t^{a.t.} = B_t$.

6. **Mark to market**

The mark to market principle collects a certain percentage T_t of *any* price change in any period. However, when the tax liabilities are left in the tax account A_t the amounts charged at any intermediate point in time u must be discounted forward to time t:

$$\begin{aligned}
A_t^S &= \int_0^t T_u \frac{B_t^{a.t.}}{B_u^{a.t.}} dS_u \\
&= T_t S_t - T_0 B_t^{a.t.} S_0 + \int_0^t r_u T_u (1 - T_u) \frac{B_t^{a.t.}}{B_u^{a.t.}} S_u du \qquad (3.23) \\
S_t - A_t^S &= S_t - \int_0^t T_u \frac{B_t^{a.t.}}{B_u^{a.t.}} dS_u \\
&= (1 - T_t) S_t + T_t B_t^{a.t.} S_0 - \int_0^t r_u T_u (1 - T_u) \frac{B_t^{a.t.}}{B_u^{a.t.}} S_u du \qquad (3.24) \\
d(S_t - A_t^S) &= r_t (1 - T_t)(S_t - A_t^S) dt + (1 - T_t) \sigma S_t (dZ_t + \lambda_t dt) \qquad (3.25)
\end{aligned}$$

The mark-to-market taxation system is characterized by a risk sharing mechanism between the investor and the government which taxes the before tax return with the same tax rate as interest income. In terms of the measures K_{tu} we have

$$K_{tu} = 1_{\{u>0\}} T_u \left[\frac{B_t^{a.t.}}{B_u^{a.t.}} \right] + (1 - T_t) 1_{\{u=t\}} \qquad (3.26)$$

When the tax rate is a constant T the mark-to-market taxation system can also be interpreted as a convex combination of the two systems:

- "government takes all risk" with weight T
- imputed wealth tax with weight $1 - T$

4 Necessary and sufficient conditions

In this section we provide necessary and sufficient conditions for any tax functional K_{tu} to be neutral. This is formulated in theorem 1 below in slightly more general terms than in the examples shown above in section 3; these examples turn out to fit into these conditions as special cases.

S_u is a vector process of asset prices and θ^u is a self-financing trading strategy. One of the assets in the vector S_u is the risk-free asset in accordance with the price process (3.5).

[3)] Sometimes referred to as the "Brown tax".

In a linear taxation system value relations based on the "no arbitrage principle" can be represented by some equivalent martingale measure Q such that appropriately discounted values are martingales. The relevant discount factor under taxation is the bank account after tax. In order for Q to be an equivalent martingale measure for a given taxable investor it must be true that for any two points in time t and s, $0 < t < s \leq H$:

$$\frac{1}{B_t^{a.t.}} \int_0^t \theta^u S_u dK_{tu} = E^Q \left[\int_0^s \theta^u S_u dK_{su} \frac{1}{B_s^{a.t.}} \Big| \mathcal{F}_t \right] \quad \Leftrightarrow \tag{4.1}$$

$$0 = E^Q \left[\int_0^t \theta^u S_u \left(dK_{su} \frac{B_t^{a.t.}}{B_s^{a.t.}} - dK_{tu} \right) + \int_t^s \theta^u S_u dK_{su} \frac{B_t^{a.t.}}{B_s^{a.t.}} \Big| \mathcal{F}_t \right] \tag{4.2}$$

The following two relations (4.3)-(4.4) turn out to be sufficient, and under an innocent regularity condition also necessary, conditions for neutrality to rule out any wash sale opportunities and timing options:

$$dK_{tu} = \frac{B_t^{a.t.}}{B_s^{a.t.}} dK_{su} \quad (0 \leq u < t) \tag{4.3}$$

$$dK_{tt} = \int_t^s \frac{B_t^{a.t.}}{B_s^{a.t.}} \frac{B_u}{B_t} dK_{su} \quad (t < s \leq H) \tag{4.4}$$

The interpretation of these conditions is clear. (4.3) states that the value of the tax levied on the value measured at time $u < t$, appropriately discounted by the relevant bank account after tax, is independent of the date t chosen for realization of the position. The value of the tax will be known and any uncertainty in this regard is resolved at time u. Any attempt to defer this tax will be fully neutralized by an appropriate increase in the amount to be paid − an increase governed by the increase in the investor's bank account after tax.

As for the "diagonal element" (4.4), consider the fact that for any self-financing trading strategy θ^u throughout the interval $[0, s]$ there is a corresponding self-financing trading strategy $\widetilde{\theta}^u$ that involves a liquidation of all risky positions at time t and a reinvestment of the proceeds in a synthetically created bank account that accumulates the interest payments over the period $[t, s]$ as capital gains. I.e.

$$\widetilde{\theta}^u S_u \equiv \begin{cases} \theta^u S^u \text{ for } 0 \leq u < t \\ \theta^t S_t \frac{B_u}{B_t} \text{ for } t \leq u \leq s \end{cases} \tag{4.5}$$

Replacing θ^u by $\widetilde{\theta}^u$ in (4.2) gives the result:

$$0 = E^Q \left[\int_0^t \theta^u S_u \left(dK_{su} \frac{B_t^{a.t.}}{B_s^{a.t.}} - dK_{tu} \right) + \int_t^s \theta^t S_t \frac{B_u}{B_t} \frac{B_t^{a.t.}}{B_s^{a.t.}} dK_{su} \Big| \mathcal{F}_t \right] \tag{4.6}$$

The integral over $[0, t)$ vanishes due to (4.3). The remaining terms amount to the statement

$$0 = \theta^t S_t E^Q \left[dK_{tt} - \frac{B_t^{a.t.}}{B_s^{a.t.}} \int_t^s \frac{B_u}{B_t} dK_{su} \Big| \mathcal{F}_t \right] \tag{4.7}$$

for which (4.4) is clearly a sufficient condition.

The economic interpretation of (4.7) can be seen as an irrelevance condition. The integral term gives the final result from investing in the synthetic bank account. Instead of the synthetic bank account the investor might have chosen to invest in an ordinary bank account with periodical taxation of interest payments. This would have produced the final result $dK_{tt}\left(B_s^{a.t.}/B_t^{a.t.}\right)$. (4.4) states that these two routes produce identical results at time s.

(4.3)-(4.4) are only relative conditions. However, by the same reasoning applied for $t=0$ we obtain the anchoring

$$B_s^{a.t.} = \int_0^s B_u dK_{su} \quad \Leftrightarrow \quad dK_{00} = \int_0^s \frac{B_u}{B_s^{a.t.}} dK_{su} = 1 \tag{4.8}$$

The implication that the difference over which the conditional expected value is taken in (4.7) must vanish pointwise as stated in (4.4) is a stronger condition than (4.7) and clearly sufficient for it to be true. It is a rather strong condition in the sense that the tax functional must produce an $\mathcal{F}_t$-measurable result from integrating the $\mathcal{F}_u$-measurable variables $(B_u/B_t)\left(B_t^{a.t.}/B_s^{a.t.}\right) dK_{su}$ over the interval $[t,s]$.

(4.3)-(4.4) determine the **intertemporal structure** of the tax functional. These conditions are the ones naturally arising in order to eliminate intertemporal tax arbitrage opportunities by investing in riskless assets. Well known mechanisms of this kind under a realization system is to create an asymmetry between the timing of tax deductible interest payments and taxable capital gains. The conditions in (4.3)-(4.4) rule out such timing options and they depend solely upon the tax rules for the bank account. There is still a choice to be made − and it is the only choice left for policy makers − with respect to the design of the **risk sharing mechanism** between the investor and the government in a neutral linear taxation system.

Theorem 1 *Linear and symmetric neutral taxation schedules in the continuous time "no arbitrage" pricing model are exhaustively characterized as follows.*

1. *Assume that Q is an equivalent martingale measure before tax. Then valuation neutrality for any given taxable investor in the sense that for the same equivalent martingale measure Q, any pair of dates (t,s), $t<s\leq H$ and any self-financing trading strategy $u \to \theta^u$:*

$$\int_0^t dK_{t,u} \frac{\theta^u S_u}{B_t^{a.t.}} = E^Q\left[\int_0^s dK_{s,u}\frac{\theta^u S_u}{B_s^{a.t.}} \,\middle|\, \mathcal{F}_t\right] \tag{4.9}$$

   is guaranteed whenever the tax functional satisfies the following conditions:

$$\begin{aligned} dK_{tt} &= \int_t^s \frac{B_t^{a.t.}}{B_s^{a.t.}}\frac{B_u}{B_t} dK_{su} \quad (t<s\leq H), \quad dK_{00}=1 && (4.10)\\ dK_{tu} &= dK_{su}\frac{B_t^{a.t}}{B_s^{a.t.}} \quad (0\leq u<t) && (4.11)\end{aligned}$$

2. *Assume that non-trivial self-financing strategies exist, i.e. strategies such that $\theta^u S_u/B_u$ is never constant on any interval, then valuation neutrality for any given taxable investor in the sense described in (4.9) requires the restrictions (4.10)-(4.11) as necessary conditions.*

3. *When $dK_{tt}\neq 0$ $\forall t$ valuation neutrality for some taxable investor with some equivalent martingale measure Q is equivalent to valuation neutrality for any other investor with the same equivalent martingale measure, but other taxation functionals satisfying (4.10)-(4.11). In particular, valuation neutrality for some investor implies that the before tax valuation relation must be fulfilled with the same martingale measure Q.*

4. *Any convex combination of two linear and symmetric neutral taxation functionals is again a linear and symmetric taxation functional.*

5. *The set of attainable claims after tax is the same for all investors for which $dK_{tt} \neq 0 \ \forall t$, and this set is identical to the set of attainable claims before tax.* ∎

Proof *Statement 4 is trivial, given the characterization in (4.10)-(4.11).*

The rest of the proof involves some lengthy calculations that are devoted to the Appendix. ∎

Theorem 2 *For any linear and symmetric neutral taxation system the wealth dynamics for an unrealized portfolio position as well as the dynamics for the tax account is fully described by the development of the bank account after tax and the diagonal elements dK_{tt} as*

$$\begin{aligned} d(\theta^t S_t - A_t^\theta) &= r_t(1-T_t)(\theta^t S_t - A_t^\theta)dt + dK_{tt}(\theta^t dS_t - r_t\theta^t S_t dt) && (4.12) \\ dA_t^\theta &= r_t(1-T_t)A_t^\theta dt + (1-dK_{tt})(\theta^t dS_t - r_t\theta^t S_t dt) + r_t T_t \theta^t S_t dt && (4.13) \end{aligned}$$

∎

Remark The dynamics in (4.12)-(4.13) can be interpreted as a combination of an imputed wealth tax ($r_t T_t \theta^t S_t dt$) and a mark-to-market tax $(1 - dK_{tt})(\theta^t dS_t - r_t\theta^t S_t dt)$ levied on the risky part of the gross amount of wealth. ∎

Proof

$$\begin{aligned} \theta^t S_t - A_t^\theta &= \int_0^t \theta^u S_u dK_{tu} \\ &= \int_0^t \theta^u S_u dK_{Tu} \frac{B_t^{a.t.}}{B_T^{a.t.}} + \theta^t S_t dK_{tt} \\ &= \int_0^t \theta^u S_u dK_{Tu} \frac{B_t^{a.t.}}{B_T^{a.t.}} + \theta^t S_t \int_t^T \frac{B_u}{B_t} \frac{B_t^{a.t.}}{B_T^{a.t.}} dK_{Tu} && (4.14) \\ d(\theta^t S_t - A_t^\theta) &= \theta^t S_t dK_{Tt} \frac{B_t^{a.t.}}{B_T^{a.t.}} + r_t(1-T_t)\int_0^t \theta^u S_u dK_{Tu}\frac{B_t^{a.t.}}{B_T^{a.t.}} + \theta^t dS_t dK_{tt} + \\ &\quad r_t(1-T_t)\theta^t S_t dK_{tt} - \theta^t S_t dK_{Tt}\frac{B_t^{a.t.}}{B_T^{a.t.}} - r_t\theta^t S_t dK_{tt}dt && (4.15) \\ &= r_t(1-T_t)(\theta^t S_t - A_t^\theta)dt + dK_{tt}(\theta^t dS_t - r_t\theta^t S_t dt) && (4.16) \\ A_t^\theta &= \theta^t S_t - \int_0^t dK_{tu}\theta^u S_u = (1-dK_{tt})\theta^t S_t - \int_0^t dK_{tu}\theta^u S_u && (4.17) \\ dA_t^\theta &= r_t(1-T_t)A_t^\theta dt + (1-dK_{tt})(\theta^t dS_t - r_t\theta^t S_t dt) + r_t T_t\theta^t S_t dt && (4.18) \end{aligned}$$

∎

Examples from section 3.

In table 1 we list the relevant diagonal element for the specific examples provided in section 3.

"Government takes all risk"	$dKtt\!=\!0$
"Imputed wealth tax"	$dKtt\!=\!1$
Auerbach (1991)	$dKtt\!=\!\frac{B_t^{a.t.}}{B_t}$
Auerbach&Bradford (2001)	$(1-H)\frac{B_t^{a.t.}}{B_t}$
"Mark-to-market"	$dK_{tt}\!=\!1\!-\!T_t$

Table 1: Tax rate for risky income for specific examples

Furthermore, we claimed that the realization principle could not be neutral. When interest rates as well as tax rates are constant over time we check this by plugging into the anchoring condition (4.8). Using Jensen's inequality[4)] we see that

$$T\frac{1}{B_t^{a.t.}}+(1-T)\frac{B_t}{B_t^{a.t.}}>1 \quad\Leftrightarrow\quad B_t^{a.t.}<T+(1-T)B_t=B_t-T(B_t-1) \tag{4.19}$$

With time varying tax rates it is not possible to say anything general about the advantage of deferring a tax liability. Assume therefore that the tax rate valid at the realization date is the tax rate valid for taxation of interest payments. Then the realization principle could be made a neutral system by adjusting for the tax rate with which the investor is credited for the original acquisition price. Call this latter tax rate T_0. Then neutrality requires

$$T_0\frac{1}{B_t^{a.t.}}+(1-T)\frac{B_t}{B_t^{a.t.}}=1 \quad\Leftrightarrow\quad T_0=B_t^{a.t.}-(1-T)B_t \tag{4.20}$$

For any time horizon $t>1$ we must have $T_0<T$. Furthermore, T_0 is a decreasing function of t in order to correct for the deferral advantage. It will eventually even become negative.

Alternatively, assume that the tax rate valid at the acquisition date is the tax rate valid for taxation of interest payments. Then neutrality requires

$$T\frac{1}{B_t^{a.t.}}+(1-T_t)\frac{B_t}{B_t^{a.t.}}=1 \quad\Leftrightarrow\quad T_t=\left(1-\frac{B_t^{a.t.}}{B_t}\right)+\frac{T}{B_t} \tag{4.21}$$

The tax rate at realization is increasing in t with an upper limit 1. As t grows it looks more and more like the Auerbach (1991) retrospective tax system, which is the first term on the rhs of (4.21). ▮

[4)] Applied to the functions 1 and B_t, respectively.

5 Portfolio choice and neutral taxation

So far the analysis has not dealt with neither a portfolio choice nor an intertemporal consumption-investment saving decision. In order to incorporate this it is necessary to specify an intertemporal utility function as well as finding the right budget dynamics for the expected utility optimizing investor.

Whenever individual assets can be combined into portfolios in a perfectly divisible manner it is an arbitrary choice what is considered as an individual asset with its own price function. For simplicity we will abandon the notation of self-financing portfolios and simply interpret S_t as a scalar process describing the price dynamics of the dynamically managed portfolio made up of basic marketed assets that the investor has chosen to invest in in order to carry consumption possibilities forward in time. This investment induces taxation consequences through the tax account A_t. Simultaneously, money is withdrawn from the portfolio – or a similar short position in the bank account is built up – in order to pay for the flow of consumption.

Let $\widehat{W}_t$ be the net wealth. Then the following sequence of calculations follows from the same principles as in theorem 2:

$$\begin{aligned}
S_t - A_t &= \int_0^t c_u \frac{B_t^{a.t.}}{B_u^{a.t.}} du + \widehat{W}_t \qquad (5.1)\\
\int_0^t S_u dK_{tu} + S_t dK_{tt} &= \int_0^t c_u \frac{B_t^{a.t.}}{B_u^{a.t.}} du + \widehat{W}_t \\
\int_0^t S_u \frac{B_t^{a.t.}}{B_T^{a.t.}} dK_{Tu} + S_t \int_t^T \frac{B_u}{B_t} \frac{B_t^{a.t.}}{B_T^{a.t.}} dK_{Tu} &= \int_0^t c_u \frac{B_t^{a.t.}}{B_u^{a.t.}} du + \widehat{W}_t \quad \Rightarrow \qquad (5.2)\\
S_t \frac{B_t^{a.t.}}{B_T^{a.t.}} dK_{Tt} + r_t(1-T_t) \left[\int_0^t S_u \frac{B_t^{a.t.}}{B_T^{a.t.}} dK_{Tu} + S_t dK_{tt} \right] & \\
-S_t \frac{B_t^{a.t.}}{B_T^{a.t.}} dK_{Tt} + (dS_t)(dK_{tt}) - r_t S_t dK_{tt} - c_t dt - d\widehat{W}_t &= r_t(1-T_t) \int_0^t c_u \frac{B_t^{a.t.}}{B_u^{a.t.}} du \qquad (5.3)
\end{aligned}$$

After cancellation of terms the budget dynamics becomes

$$d\widehat{W}_t = r_t(1-T_t)\widehat{W}_t dt + dK_{tt}(dS_t - r_t S_t dt) - c_t dt \qquad (5.4)$$

This is a pure budget equation that involves no optimization.

First we rewrite the dynamics for S_t in a standard manner in terms of the risk premium λ_t:

$$dS_t = (r_t + \lambda_t \sigma_t) S_t dt + \sigma_t S_t dZ_t \quad \Leftrightarrow \quad dS_t - r_t S_t dt = \sigma_t S_t \left[\lambda_t dt + dZ_t \right] \qquad (5.5)$$

Second, we insert this into the budget dynamics to arrive at

$$d\widehat{W}_t = r_t(1-T_t)\widehat{W}_t dt + \sigma_t S_t(\lambda_t dt + dZ_t) - c_t dt \qquad (5.6)$$

This shows that the investor faces two choices:

1. How much to consume, i.e. the rate of consumption c_t must be determined.

2. The distribution of wealth into a riskless position and into risky assets, respectively; or, alternatively stated, a determination of how much volatility to bring into the wealth dynamics

Next consider the standard Merton problem, cf. Merton (1971), with a utility optimizing agent. Preferences are given by

$$\text{Max } E_0 \left[\int_0^H U(c_s, s) ds + B(\widehat{W}(H), H) \right] \tag{5.7}$$

and the investor must choose the consumption policy C_s along with a portfolio strategy to maximize her expected lifetime utility.

The HJB equation becomes

$$0 = \max_{\{\sigma_t S_t, c_t\}} \Big[U(c_t, t) + J_t + J'_{\widehat{W}}(\widehat{W}_t, t) \left[\widehat{W}_t r_t (1 - T_t) + \sigma_t S_t dK_{tt} \lambda_t - c_t dt \right] + \frac{1}{2} J''_{\widehat{W}\widehat{W}}(\widehat{W}_t, t) \left(dK_{tt} \sigma_t S_t \right)^2 \Big] \tag{5.8}$$

The usual first order conditions become

$$\begin{aligned} U'(c_t, t) &= J'_{\widehat{W}}(\widehat{W}_t, t) && (5.9) \\ 0 &= J'_{\widehat{W}}(\widehat{W}_t, t) dK_{tt} \lambda_t + J''_{\widehat{W}\widehat{W}}(\widehat{W}_t, t) (dK_{tt})^2 \sigma_t S_t \qquad \Leftrightarrow \\ \sigma_t S_t &= -\frac{J'_{\widehat{W}}(\widehat{W}_t, t)}{J''_{\widehat{W}\widehat{W}}(\widehat{W}_t, t)} \frac{\lambda_t}{dK_{tt}} \equiv \frac{1}{RRA(W_t, t)} \frac{\lambda_t}{dK_{tt}} \widehat{W}_t && (5.10) \end{aligned}$$

Compared with the usual solution to the Merton problem this can be reformulated in order to express the investment in risky assets as a fraction of investor's wealth. To compare with the traditional solution we interpret the dynamics for S_t as having σ_t fixed whereas the investor can choose the amount of money S_t to be invested in this risky portfolio. This gives the fraction of risky investment to net wealth as

$$\frac{S_t}{\widehat{W}_t} = -\frac{J'_{\widehat{W}}(\widehat{W}_t, t)}{J''_{\widehat{W}\widehat{W}}(\widehat{W}_t, t) \widehat{W}_t} \frac{\lambda_t}{\sigma_t dK_{tt}} \equiv \frac{1}{RRA(W_t, t)} \frac{\lambda_t}{\sigma_t dK_{tt}} \tag{5.11}$$

There is one obvious modification coming from the risk sharing mechanism. The position in risky assets is enlarged by the factor $1/dK_{tt}$. Considering mark-to-market taxation as a benchmark case where $dK_{tt} = 1 - T$ this means that the position in risky assets is enlarged by the factor $1/(1-T)$.

In order to see the effect on the optimal consumption policy we again revert to a well-known specification of preferences via an isoelastic utility function and isoelastic bequest function:

$$U(C_s, s) = e^{-\rho s} \frac{1}{1-\gamma} C_s^{1-\gamma}, \qquad B(W(H), H) = K e^{-\rho H} \frac{1}{1-\gamma} W(H)^{1-\gamma} \tag{5.12}$$

and pursue the usual "guess and verify" solution strategy.

Assume that the value function is of the form

$$J(\widehat{W}_t, t) = e^{-\rho t} \Psi(t)^{\gamma} \frac{1}{1-\gamma} W^{1-\gamma} \tag{5.13}$$

with partial derivatives

$$\begin{aligned} J_t'(\widehat{W}_t,t) &= (\gamma(\dot{\Psi(t)}/\Psi(t))-\rho)J && (5.14)\\ J_{\widehat{W}}'(\widehat{W}_t,t) &= (1-\gamma)(J/\widehat{W}_t) && (5.15)\\ J_{\widehat{W},\widehat{W}}''(\widehat{W}_t,t) &= (-\gamma)(1-\gamma)(J/\widehat{W}_t^2) && (5.16) \end{aligned}$$

Then the optimal consumption policy becomes

$$e^{-\rho t}C_t^{-\gamma} = e^{-\rho t}\Psi(t)^{\gamma}\widehat{W}_t^{-\gamma} \quad \Rightarrow \quad C_t = \Psi(t)^{-1}\widehat{W}_t \tag{5.17}$$

and the optimal portfolio position becomes

$$\sigma_t S_t = \frac{1}{\gamma}\frac{\lambda_t}{dK_{tt}}\widehat{W}_t \tag{5.18}$$

Inserting back into the HJB equation we get

$$\begin{aligned} 0 &= J\Psi(t)^{-1} + (\gamma(\dot{\Psi(t)}/\Psi(t))-\rho)J + \\ &\quad (1-\gamma)J\left[r_t(1-T_t)+\frac{\lambda_t^2}{\gamma}-\Psi(t)^{-1}\right]-\frac{1}{2}(1-\gamma)J\frac{\lambda_t^2}{\gamma} && (5.19) \end{aligned}$$

Observe that the taxation $(1-dK_{tt})$ of risky income disappears. This tax rate could well be different from the tax rate T_t applicable to riskless interest payments. However, it significantly affects the portfolio position as shown in (5.18); the investor takes a position that is suited to neutralize the effect of taxation in the risky part of the wealth dynamics. The higher the tax rate the higher the fraction of wealth allocated to the risky asset. This is due to the fact that

- in a neutral taxation system the risk premium λ is unaffected by taxation. For the simplest case, the mark-to-market taxation system with a constant tax rate T, the expected return as well as the risk-free rate of interest is multiplied by $1-T$ in the numerator, but so is the volatility parameter σ in the denominator. The government takes a share of the expected return on any asset, but simultaneously also shares the risk with the investor.
- the volatility parameter in the usual before tax term λ/σ in the asset demand function is multiplied by $1-T$

After cancellation of terms we end up with an HJB equation very similar to the well-known one from the case without taxation. The only difference is that the riskless rate of interest is now the after tax rate $r_t(1-T_t)$:

$$\begin{aligned} \gamma\dot{\Psi(t)} &= \left[(1-\gamma)\left(r_t(1-T_t)+\frac{1}{2}\frac{\lambda_t^2}{\gamma}\right)-\rho\right]\Psi(t)+\gamma \quad \Leftrightarrow \\ \dot{\Psi(t)} &= -a_t\Psi(t)+1 && (5.20)\\ a_t &\equiv \frac{\rho-(1-\gamma)\left(r_t(1-T_t)+\frac{1}{2}\frac{\lambda_t^2}{\gamma}\right)}{\gamma} && (5.21) \end{aligned}$$

with the obvious boundary condition $\Psi(H)^{\gamma}=K$. With constant parameters the solution to (5.20)-(5.21) is

$$\Psi(t) = \frac{1-e^{a(t-H)}(1-aK^{1/\gamma})}{a} \tag{5.22}$$

Theorem 3 *For the classical Merton (1971) lifetime consumption-investment problem with isoelastic utility and bequest functions and constant investment opportunity set and tax rates the following is true:*

1. *The consumption policy as a fraction of net wealth is identical to the classical solution with the riskless rate of interest substituted by its after-tax counterpart rate of interest:*

$$c_u = \Psi(t)^{-1}\widehat{W}_u \tag{5.23}$$

where

$$\Psi(t)^{-1} = \frac{a}{1-e^{a(t-H)}(1-aK^{1/\gamma})}, \quad a = \frac{\rho-(1-\gamma)\left(r(1-T)+\frac{1}{2}\frac{\lambda^2}{\gamma}\right)}{\gamma} \tag{5.24}$$

The taxation of risky income is irrelevant.

2. *The net wealth follows a lognormal process with time-varying parameters:*

$$\widehat{W}_t = \widehat{W}_0 \exp^{\left(\frac{r(1-T)+\frac{\lambda^2}{2}-\rho}{\gamma}\right)t+\frac{\lambda}{\gamma}Z_t}\frac{1-e^{-a(H-t)}(1-aK^{1/\gamma})}{1-e^{-aH}(1-aK^{1/\gamma})} \tag{5.25}$$

Hence, it stays positive under all circumstances.

3. *The lhs of (5.1) is given by*

$$S_t - A_t = \widehat{W}_0 e^{r(1-T)t} + \int_0^t \frac{\lambda^2}{\gamma}e^{r(1-T)(t-u)}\widehat{W}_u du + \int_0^t \frac{\lambda}{\gamma}e^{r(1-T)(t-u)}\widehat{W}_u dZ_u \tag{5.26}$$

4. *The consumption process on the rhs of (5.1) is*

$$\int_0^t c_u e^{r(1-T)(t-u)}du = \int_0^t \frac{ae^{r(1-T)(t-u)}\widehat{W}_u}{1-(1-aK^{1/\gamma})e^{a(t-H)}}du \tag{5.27}$$

6 Summary and conclusion

A Appendix

Proof (Theorem 1)[5)]

We first prove statement 1. Consider (4.11). From this relation it follows that

$$\frac{dK_{tu}}{B_t^{a.t.}} = \frac{dK_{su}}{B_s^{a.t.}} \in \mathcal{F}_t \; \forall u < t \tag{A.1}$$

[5)] The proof in this version is incomplete and only a sketch that needs further mathematical checks.

Hence, letting $t \downarrow u$ *it follows from the standard right continuity condition of filtrations that*

$$\frac{dK_{su}}{B_s^{a.t.}} \in \mathcal{F}_u \ \forall s > u$$

When Q *is an equivalent martingale measure before tax and (4.10) is fulfilled we get the desired result from the following sequence of calculations:*

$$\frac{\theta^t S_t}{B_t} = E^Q\left[\frac{\theta^s S_s}{B_s}\,\middle|\,\mathcal{F}_t\right] \quad \wedge \quad dK_{tt} = \int_t^s \frac{B_t^{a.t.}}{B_s^{a.t.}}\frac{B_u}{B_t} dK_{su} \quad (\in \mathcal{F}_t) \quad \Rightarrow \tag{A.2}$$

$$\begin{aligned}
\frac{\theta^t S_t}{B_t^{a.t.}} dK_{tt} &= E^Q\left[\frac{\theta^s S_s}{B_s}\int_t^s \frac{B_u}{B_s^{a.t.}} dK_{su}\,\middle|\,\mathcal{F}_t\right] \\
&= E^Q\left[\int_t^s E^Q\left[\frac{\theta^s S_s}{B_s}\frac{B_u}{B_s^{a.t.}} dK_{su}\,\middle|\,\mathcal{F}_u\right]\,\middle|\,\mathcal{F}_t\right] \\
&= E^Q\left[\int_t^s E^Q\left[\frac{\theta^s S_s}{B_s}\,\middle|\,\mathcal{F}_u\right]\frac{B_u}{B_s^{a.t.}} dK_{su}\,\middle|\,\mathcal{F}_t\right] \\
&= E^Q\left[\int_t^s \frac{\theta^u S_u}{B_s^{a.t.}} dK_{su}\,\middle|\,\mathcal{F}_t\right] \qquad \text{(A.3)}
\end{aligned}$$

Using (4.11) the terms in $[0,t)$ *can just be added on to both sides of (A.3) to obtain (4.9):*

$$\int_0^t \frac{\theta^u S_u}{B_t^{a.t.}} dK_{tu} = E^Q\left[\int_0^s \frac{\theta^u S_u}{B_s^{a.t.}} dK_{su}\,\middle|\,\mathcal{F}_t\right] \tag{A.4}$$

This completes the proof of 1.

To prove 3. we reverse the argument. Assume that $\forall t \in (0,s)$ *and for all self-financing trading strategies* θ^u *we know that*

$$\frac{\theta^t S_t}{B_t}\int_t^s \frac{B_u}{B_s^{a.t.}} dK_{su} = E^Q\left[\int_t^s \frac{\theta^u S_u}{B_u}\frac{B_u}{B_s^{a.t.}} dK_{su}\,\middle|\,\mathcal{F}_t\right] \tag{A.5}$$

We also know that

$$0 \neq \int_t^s \frac{B_u}{B_s^{a.t.}} dK_{su} \ (\in \mathcal{F}_t) \tag{A.6}$$

Then

$$E^Q\left[\int_t^s \left(\frac{\theta^t S_t}{B_t} - \frac{\theta^u S_u}{B_u}\right)\frac{B_u}{B_s^{a.t.}} dK_{su}\,\middle|\,\mathcal{F}_t\right] = 0 \quad \Leftrightarrow$$

$$E^Q\left[\int_t^s \left(\frac{\theta^u S_u}{B_u} - \frac{\theta^t S_t}{B_t}\right)\frac{B_t^{a.t.}}{B_t}\frac{B_u}{B_s^{a.t.}} dK_{su}\,\middle|\,\mathcal{F}_t\right] = 0 \tag{A.7}$$

The goal is to prove that $\frac{\theta^t S_t}{B_t}$ *is an* $\mathcal{F}_t$*-martingale. In order to proceed the statement (A.7) can be rewritten, using Fubini's theorem for stochastic integrals, as follows:*

$$E^Q\left[\int_t^s \left[\int_t^u d\left(\frac{\theta^v S_v}{B_v}\right)\right] \frac{B_t^{a.t.}}{B_t}\frac{B_u}{B_s^{a.t.}} dK_{su} \,\middle|\, \mathcal{F}_t\right] \;=\; 0 \;\Leftrightarrow \tag{A.8}$$

$$E^Q\left[\int_t^s \left[\int_v^s \frac{B_t^{a.t.}}{B_t}\frac{B_u}{B_s^{a.t.}} dK_{su}\right] d\left(\frac{\theta^v S_v}{B_v}\right) \,\middle|\, \mathcal{F}_t\right] \;=\; 0 \;\Leftrightarrow \tag{A.9}$$

$$E^Q\left[\int_t^s d\left(\frac{\theta^v S_v}{B_v}\right) dK_{vv} \,\middle|\, \mathcal{F}_t\right] \;=\; 0 \tag{A.10}$$

This also shows that

$$E^Q\left[\int_0^s d(\frac{\theta^v S_v}{B_v}) dK_{vv} \,\middle|\, \mathcal{F}_t\right] = \int_0^t d\left(\frac{\theta^v S_v}{B_v}\right) dK_{vv} \tag{A.11}$$

Hence, $\int_0^s d\left(\frac{\theta^v S_v}{B_v}\right) dK_{vv}$ *is an* $\mathcal{F}_t$*-martingale. If*

$$d\left(\frac{\theta^v S_v}{B_v}\right) = \mu^\theta(S_t,t)dt + \sigma^\theta(S_t,t)dZ_t \tag{A.12}$$

then the statement in (A.11) means that

$$\int_0^s \mu^\theta(S_v,v)dK_{vv} \tag{A.13}$$

is an $\mathcal{F}_t$*-martingale. At the same time it is also continuous; hence it is constant. Since this constant must be zero one can also differentiate after* s *and use the assumption that* $dK_{ss} \neq 0$ *to conclude that* $\mu^\theta(S_s,s) = 0$.

This completes the proof of statement 3., since the ability to extract a before tax relation together with 1. ensures that one can always establish the after tax relations for any other investor. The restriction $dK_{tt} \neq 0$ *essentially rules out a taxation principle like "government takes all risk", where it is obviously not possible to "calculate backwards".*

The proof of 2. is technically more involved (and not yet included).

Finally, to prove 5. we rely on the relations (4.12)-(4.13) derived in theorem 2, but independent of the claim 5. Let Ψ^t *be a self-financing portfolio strategy replicating a contingent claim. Then:*

$$\Psi^t dS_t \;=\; r_t\Psi^t S_t dt + \Psi^t(dS_t - r_t S_t dt) = r_t\Psi^t S_t dt + \lambda_t\sigma_t^\Psi \Psi^t S_t dt + \sigma_t^\Psi \Psi^t S_t dZ_t \;\Rightarrow \tag{A.14}$$

$$\Psi^t S_t \;=\; B_t\Psi^0 S_0 + \int_0^t \sigma_u^\Psi \Psi^u S_u \frac{B_t}{B_u}(dZ_u + \lambda_u du) \tag{A.15}$$

A self-financing portfolio strategy θ^t *will produce the following result after tax:*

$$\theta^t S_t - A_t^\theta = B_t^{a.t.}\theta^0 S_0 + \int_0^t dK_{uu}\frac{B_t^{a.t.}}{B_u^{a.t.}}\sigma_u^\theta \theta^u S_u(dZ_u + \lambda_u du) \tag{A.16}$$

With a given strategy Ψ^t *we first adjust the strategy* θ^u *such that the value of the integral is the same. The recipe for this is given by*

$$\theta^u = \left(\frac{B_u^{a.t.}}{B_t^{a.t.}}\frac{B_t}{B_u}\frac{1}{dK_{uu}}\right)\Psi^u \tag{A.17}$$

Since $dK_{00} \equiv 1$ *this is also true for* $u = 0$ *and results in identical values also for the first term.* ▮

References

AUERBACH, A. J. (1991): "Retrospective capital gains taxation," *American Economic Review*, 81(1):167–178.

AUERBACH, A. J. AND D. BRADFORD (2001): "Generalized cash flow taxation," Working Paper 8122, National Bureau of Economic Research.

BRADFORD, D. F. (1995): "Fixing realization accounting," *Tax Law Review*, 53(1):345–376. Reprinted as chapter 14 in Bradford (2000).

——— (2000): *Taxation, wealth and saving,* MIT Press, Cambridge, Massachusetts.

BROWN, E. C. (1948): "Business income, taxation, and investment incentives," in Metzler, L. A. et. al. (1948).

CADENILLAS, A. AND S. R. PLISKA (1999): "Optimal trading of a security when there are taxes and transactions costs," *Finance and Stochastics*, 3:137–165.

CHEN, A. (1987): *Research in Finance,* JAI Press, Greenwich, CT.

CONSTANTINIDES, G. M. (1983): "Capital market equilibrium with personal taxes," *Econometrica*, 51:611–636.

CONSTANTINIDES, G. M. AND J. INGERSOLL (1984a): "Optimal bond trading with personal taxes," *Journal of Financial Economics*, 13:299–335.

——— (1984b): "Optimal stock trading with personal taxes: Implications for prices and the abnormal January return," *Journal of Financial Economics*, 13:65–89.

DAMMON, R. M. (1987): "A security market and capital structure equilibrium under uncertainty with progressive personal taxes," in *Chen (1987)*, pages 53–74.

DAMMON, R. M. AND R. C. GREEN (1987): "Tax arbitrage and the existence of equilibrium prices for financial assets," *The Journal of Finance*, 42:1143–1166.

DAMMON, R. M. AND C. S. SPATT (1996): "The optimal trading and pricing of asymmetric capital gains taxes and transaction costs," *The Review of Financial Studies*, 9:921–952.

DERMODY, J. C. AND E. Z. PRISMAN (1988): "Term structure multiplicity and clientele in markets with transactions costs and taxes," *The Journal of Finance*, 41:893–911.

DERMODY, J. C. AND R. T. ROCKAFELLER (1991): "Cash stream valuation in the face of transactions costs and taxes," *Mathematical Finance*, 1(1):31–54.

——— (1995): "Tax basis and nonlinearity in cash stream valuation," *Mathematical Finance*, 5(2):97–119.

DYBVIG, P. AND S. ROSS (1986): "Tax clienteles and asset pricing," *The Journal of Finance*, 41:751–763.

IKEDA, N. AND S. WATANABE (1989): *Stohastic Differential Equations and Diffusion Processes,* volume 24 of *North-Holland Mathematical Library.* North-Holland Publishing Company/Kodansha Ltd., Amsterdam, The Netherlands/Tokyo, Japan, 2nd edition.

Merton, R. C. (1971): "Optimum consumption and portfolio rules in a continuous time model," *Journal of Economic Theory*, 3:373–413. Erratum: Merton (1973). Reprinted in Merton (1992, Chapter 5).

——— (1973): "Erratum," *Journal of Economic Theory*, 6:213–214.

——— (1992): *Continuous-Time Finance,* Basil Blackwell Inc., Cambridge, Massachusetts.

Metzler, L. A. et. al., editor (1948): *Income, Employment and Public Policy: Essays in Honor of A.H. Hansen,* W.W. Norton, New York, USA.

Ross, S. A. (1987): "Arbitrage and martingales with taxation," *Journal of Political Economy*, 95:371–93.