Attainability of European Path-Independent Claims in Incomplete Markets

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Abstract

In this paper we consider the question which path-independent claims are attainable through self-financing trading strategies in an incomplete market. For continuous-time stochastic volatility models we show that from this special group of claims only affine linear payoffs can be replicated. For models exhibiting stochastic interest rates as a second source of risk in addition to the stock price the class of attainable path-independent payoffs is even smaller, since only multiples of the terminal stock can be replicated.

We provide a simple proof for this proposition based on the requirement that, for replication, the stock and the claim must be locally perfectly correlated, and based on the partial differential equation that any path-independent claim has to satisfy. An important application of our result is the quick derivation of bounds on European option prices which were previously deduced by other authors using very demanding techniques from probability theory. Furthermore, we show that there is no analogy for our result in models with discrete time and discrete state variables, i.e. in these models we can generate at least some non-linear path-independent claims by self-financing trading strategies.

Keywords: Incomplete markets, attainability, stochastic volatility, stochastic interest rates, superhedging

JEL: G13

1 Introduction and Motivation

A market is called incomplete, if there exist payoff patterns that cannot be replicated by self-financing dynamic trading strategies. An immediate consequence of market incompleteness is that not all derivative assets can be valued using the standard pricing techniques based on the duplication principle. For example, it is well-known that in an economy where the stock price exhibits stochastic volatility (SV) or jumps of random sizes it is impossible to uniquely price a European call option using only no arbitrage considerations. Merton (1973) derives no arbitrage bounds for the price of a European call option based on the fact that the payoff of the option is lower than the payoff of the stock itself and greater than the payoff of the stock minus the strike price. In contrast to the call these payoffs are actually attainable, yielding the well-known price bounds. Frey and Sin (1999) have shown that these bounds are even the tightest ones for a large class of SV models. Thus, the only attainable payoff dominating the call is the stock price itself.

This strict result raises the important questions whether it can be generalized to other stochastic settings and other types of derivatives and, even more fundamentally, which types of path-independent payoffs are attainable at all. It is clear that European-type payoffs which are affine linear in the terminal stock price can always be generated by a simple buy-and-hold strategy in the stock and a default-free discount bond. If interest rates are stochastic but no bonds are traded, like in the short rate model of Cox, Ingersoll, and Ross (1985), the set of trivially attainable path-independent claims reduces even further to those payoffs that are linear in the terminal stock price. Then, although it is clear that *not all* payoffs are attainable in an incomplete market, one would intuitively think that due to the huge set of trading strategies it should be possible to generate at least *some* path-independent payoffs which are not linear in the terminal stock price. In contrast to this intuition, however, we will prove as our main result in this paper, that the set of these additional attainable path-independent payoffs is empty for a wide class of commonly used continuous-time option pricing models. We show this result for models with SV, like those introduced by Heston (1993) or Schöbel and Zhu (2000) and also for models which contain a stochastic short rate as a second source of risk besides the stock price.

Besides this fundamental result per se another important aspect of our analysis in this paper is that we offer a straightforward proof without the complicated machinery of continuous-time probability theory as it is employed for example by Eberlein and Jacod (1997) and by Frey and Sin (1999). We focus on path-independent claims only. The intuition behind our results is based on the fact that a claim can be replicated using just the stock and the money market account if and only if the claim and the stock are locally perfectly correlated. This can be easily seen in the binomial model where any terminal payoff can be generated via dynamic self-financing strategies, since it is necessarily affine linear in the stock over any one-period submodel. The main question is then which restrictions this locally perfect correlation imposes on the global characteristics of the claim, and here especially on its payoff in the case of path-independence.

In continuous-time models the perfect local correlation with the stock can only hold if the price dynamics of the claim depend on the dynamics of the stock, but not on the stochastic behavior of any nontraded risk factors, like SV. This is trivially true in the Black-Scholes model. In the presence of stochastic jumps the requirement of perfect local correlation translates into the requirement that the claim price reacts linearly to a jump in the stock price, as shown by Grünewald (1998). Imposing this independence restriction in the SV model the stochastic differential equation for the claim price shows that this price must not depend on volatility. Nevertheless the claim could still represent a basically arbitrary function of the stock price. However, if the claim is indeed independent of volatility the partial differential equation implies that the value of the claim must be an affine linear function of the current stock price. If we apply an analogous technique to models with a stochastic short rate where only the stock and the money market account are traded we find the even stronger result that only linear (as opposed to *affine* linear) payoffs are traded.

Our results have important applications. Besides offering simpler proofs than those shown in previous papers we are also able to generalize the results derived by Frey and Sin (1999) and by Frey (2000) concerning the bounds of European option prices. In particular our findings indicate that the results derived by these authors actually represent special cases of the more fundamental property of incomplete markets deduced in this paper.

Models with discrete time and discrete state variables are often used as simple but powerful analogies to continuous-time models. Concerning the issues analyzed in this paper, however, we will show that the analogies between the two classes of models are limited. Whereas it is impossible to generate nonlinear path-independent payoffs via self-financing trading strategies in the familiar continuous-time models, we can very well construct a path-independent payoff that is a nonlinear function of the terminal stock price in discrete models and which is still replicable. This will be demonstrated by means of a simple example. The intuitive reason for this result is that in a discrete model the number and range of states that can be reached at the end of the period is always finite, so that we can construct 'locally linear' payoff patterns which in some cases also aggregate to a nonlinear terminal cash flow pattern. However, in this framework we must check every payoff pattern individually for its attainability.

The rest of the paper is organized as follows. In section 2 we first describe the diffusion setup with stochastic volatility and the stochastic interest rate model. We then derive the main results concerning the attainability of path-independent claims in continuous-time models with incomplete markets. By means of a counterexample we demonstrate that this property of continuous-time models does not necessarily carry over to discrete models, constituting an important discrepancy between the two types of models. In section 3 we present the derivation of option price bounds as an important application of our results. Section 4 contains some concluding remarks.

2 Path-Independent Claims on Incomplete Markets

In this section we will derive our main result, namely that in continuous-time models with incomplete markets only those path-independent payoffs are attainable that are affine linear in S_T. We focus on SV models first and afterwards apply our line of argument also to economies with a stochastic short rate of interest (SI). Jump models are not treated in this paper, since the fundamental attainability result has already been shown by Grünewald (1998). However, the proofs for the case of SV and SI require an additional step which is not necessary in the case of jumps, as will become clear below.

2.1 Stochastic volatility

Consider the SV model given by the stochastic differential equations

$$\begin{aligned} dS_t &= \mu_S(t, S_t, V_t) S_t dt + V_t S_t dW_t^S && (1) \\ dV_t &= \mu_V(t, S_t, V_t) dt + \sigma_V(t, S_t, V_t) dW_t^V, && (2) \end{aligned}$$

where dW_t^S and dW_t^V are correlated with correlation coefficient $\rho \in (-1, +1)$. Volatility is assumed to be non-traded, so that the market is incomplete. Note that here the restriction on ρ is important, since in the degenerate case $|\rho| = 1$ the market would again be complete with only one linear independent source of risk. The market price of volatility risk, λ_V, is assumed to be a function of the current stock price, the current level of volatility V_t, and time, i.e. $\lambda_V \equiv \lambda_V(t, S_t, V_t)$. Furthermore, we assume that V can take on any positive value (or, more generally, is unbounded), in contrast to the assumptions made by Avellaneda, Levy, and Parás (1995) and by Frey and Sin (1999). Finally, we assume a deterministic interest $r \geq 0$.

In this setting we obtain the following theorem on the attainability of path-independent payoffs:

Theorem 1 *In the stochastic volatility model given by (1) and (2) a claim with terminal payoff $f(S_T)$ can be replicated by a portfolio of the stock and the money market account, if and only if the payoff is affine linear in S_T, i.e. $f(S_T) = \alpha S_T + \beta$ for some real constants α and β.*

Proof: Clearly, any affine linear payoff $f(S_T) = \alpha S_T + \beta$ can be replicated by a buy-and-hold strategy consisting of α units of the stock and an investment of βe^{-rT} into the money market account. So the 'if' part is proved.

To see the converse, note that if a payoff is traded, there exists a replicating portfolio. In order to determine this portfolio, we proceed as follows. The price of the payoff $f(S_T)$ is written as $C \equiv C(t, S_t, V_t)$. Applying Itô's lemma yields the following stochastic differential equation for the evolution of the claim price:

$$
\begin{aligned}
dC_t &= \frac{\partial C_t}{\partial t}\, dt + \frac{\partial C_t}{\partial S_t}\, dS_t + \frac{\partial C_t}{\partial V_t}\, dV_t \\
&\quad + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\, V_t^2 S_t^2\, dt + \frac{1}{2}\frac{\partial^2 C_t}{\partial V_t^2}\, \sigma_V^2\, dt + \frac{\partial^2 C_t}{\partial S_t \partial V_t}\, V_t S_t \sigma_V \rho\, dt,
\end{aligned}
$$

where the dependence of the coefficient functions on the state variables has been suppressed to save notation. In order to replicate a claim following these dynamics we need two assets, at least one of which has to depend on volatility. The usual procedure is to hedge the stochastic terms dS_t and dV_t using the two instruments and to make the strategy self-financing by using the money market account for the remaining investment (positive or negative). For a claim to be hedgeable using the stock and the money market account only, it has to exhibit a zero sensitivity with respect to volatility, i.e.

$$
\frac{\partial C(t, S_t, V_t)}{\partial V_t} = 0 \tag{3}
$$

for all t and all possible values of S_t and V_t. Thus the pricing function can be written as

$$
C(t, S_t, V_t) \equiv C(t, S_t).
$$

So the price of the claim must not depend on V, while it might still basically be an arbitrary function of the stock price. To see that the claim indeed has to be an affine linear function of S, consider the fundamental partial differential equation satisfied by the price of any path-independent claim:

$$
\begin{aligned}
&\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t}\, r S_t + \frac{\partial C_t}{\partial V_t}\, (\mu_V - \lambda_V \sigma_V) \\
&\quad + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\, V_t^2 S_t^2 + \frac{1}{2}\frac{\partial^2 C_t}{\partial V_t^2}\, \sigma_V^2 + \frac{\partial^2 C_t}{\partial S_t \partial V_t}\, V_t S_t \sigma_V \rho = r C_t,
\end{aligned} \tag{4}
$$

which, given (3), can be shortened to

$$
\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t}\, r S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\, V_t^2 S_t^2 = r C_t, \tag{5}
$$

since the second derivative with respect to volatility and the mixed term vanish. This differential equation has to hold for any two values of volatility $V_t^{(1)}$ and $V_t^{(2)}$ with $V_t^{(1)} \neq V_t^{(2)}$, so that the pricing function C has to satisfy

$$\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t}\, rS_t + \frac{1}{2}\,\frac{\partial^2 C_t}{\partial S_t^2}\,\left(V_t^{(i)} S_t\right)^2 \;=\; rC_t \tag{6}$$

for $i = 1, 2$. Subtracting (6) for $V_t = V_t^{(2)}$ from the corresponding equation for $V_t = V_t^{(1)}$ yields the following condition:

$$\frac{1}{2}\,\frac{\partial^2 C_t}{\partial S_t^2}\,\left\{\left(V_t^{(1)}\right)^2 - \left(V_t^{(2)}\right)^2\right\} S_t^2 \;=\; 0,$$

where we have used that C and therefore also its partial derivatives do not depend on V. So it must be true that

$$\frac{\partial^2 C_t}{\partial S_t^2} \;=\; 0. \tag{7}$$

This means that the pricing function is linear in the stock price. Integrating (7) once gives

$$\frac{\partial C_t}{\partial S_t} \;=\; b(t),$$

where $b(t)$ is at most a deterministic function of time. Integrating once more, we obtain

$$C_t \;=\; a(t) + b(t)S_t, \tag{8}$$

where $a(t)$ is again at most a deterministic function of time. The value of this claim at time $u < t$ is given by

$$C_u \;=\; a(t)e^{-r(t-u)} + b(t)S_u, \tag{9}$$

which follows from the fact that it can be replicated via a buy-and-hold-strategy. On the other hand, we know from (8) that C_u can also be written as

$$C_u \;=\; a(u) + b(u)S_u. \tag{10}$$

Combining (9) and (10) yields $a(u) = a(t)e^{-r(t-u)}$ and $b(t) = b(u)$, so that we end up with $a(t) = a(0)e^{rt}$ and $b(t) = b(0)$ for all t. Thus, if the payoff $f(S_T)$ can be replicated, it is of the form $f(S_T) = a(0)e^{rT} + b(0)S_T$. The duplicating portfolio consists of a simple buy-and-hold strategy with $a(0)$ units of the money market account and $b(0)$ units of the stock. □

Note that the result from Theorem 1 obviously also holds in more complex models which include jumps of random size in addition to SV. Once the market is incomplete due to SV, the introduction of further nontraded risk factors cannot augment the set of path-independent claims.

As noted above a result similar to ours from Theorem 1 was already derived for a deterministic volatility model with stochastic jumps by Grünewald (1998). However, in such an economy it is sufficient to look at the stochastic differential equation for the claim price only, since one can immediately conclude from there that the claim price has to be affine linear in S_t. As seen above, in an SV economy this argument only allows to conclude that the claim price does not depend on volatility, but it is not immediately clear that it is also affine linear in the stock price. To obtain the final result an extra step based on the analysis of the partial differential equation is necessary.

2.2 Stochastic interest rates

Consider now a model where the short term interest rate is stochastic, while the volatility of stock price changes is deterministic. This SI model is represented by the stochastic differential equations

$$dS_t = \mu_S(t, S_t, R_t)S_t dt + \sigma_S(t, S_t)S_t dW_t^S \tag{11}$$

$$dR_t = \mu_R(t, S_t, R_t)dt + \sigma_R(t, R_t)dW_t^R, \tag{12}$$

where dW_t^S and dW_t^R are correlated with correlation coefficient $\rho \in (-1, +1)$. We assume a classical short rate model, i.e. only the money market account (with a value at time t given by K_t) and the stock are traded so that the market is incomplete. The market price of interest rate risk, λ_R, is assumed to be a function of the current stock price, the current short rate R_t, and time, i.e. $\lambda_R \equiv \lambda_R(S_t, R_t, t)$.

In this setting we obtain the following theorem on the attainability of path-independent payoffs:

Theorem 2 *In a stochastic short rate model represented by (11) and (12) a claim with terminal payoff $f(S_T)$ can be replicated by a portfolio of the stock and the money market account, if and only if the payoff is linear in S_T, i.e. $f(S_T) = \alpha S_T$ for some real constant α.*

Proof: The proof will proceed along the same lines as the one shown above for Theorem 1. Clearly, any linear payoff $f(S_T) = \alpha S_T$ can be replicated by a buy-and-hold strategy consisting of α units of the stock. So the 'if' part is proved.

To see the converse, note that if a payoff is traded, there exists a replicating portfolio. The price of the payoff $f(S_T)$ is written as $C \equiv C(t, S_t, R_t)$. With Itô we obtain

$$\begin{aligned} dC_t &= \frac{\partial C_t}{\partial t}\,dt + \frac{\partial C_t}{\partial S_t}\,dS_t + \frac{\partial C_t}{\partial R_t}\,dR_t \\ &\quad + \frac{1}{2}\,\frac{\partial^2 C_t}{\partial S_t^2}\,\sigma_S^2 S_t^2\,dt + \frac{1}{2}\,\frac{\partial^2 C_t}{\partial R_t^2}\,\sigma_R^2\,dt + \frac{\partial^2 C_t}{\partial S_t \partial R_t}\,\sigma_S S_t \sigma_R \rho\,dt, \end{aligned}$$

where the dependence of the coefficient functions on the state variables has been suppressed to save notation. For a claim to be hedgeable using only the stock and the money market account, it has to exhibit a zero sensitivity with respect to interest rates, i.e.

$$\frac{\partial C(t, S_t, R_t)}{\partial R_t} = 0 \tag{13}$$

for all t and all possible values of S_t and R_t. Note that the money market account cannot be used to hedge interest rate risk, since it is locally deterministic with dynamics given by $dK_t = R_t\,K_t\,dt$.

Thus the pricing function can be written as

$$C(t, S_t, R_t) \equiv C(t, S_t).$$

The price of any path-independent claim solves the fundamental partial differential equation

$$\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t}\, R_t S_t + \frac{\partial C_t}{\partial R_t}\, (\mu_R - \lambda_R \sigma_R)$$
$$+ \frac{1}{2}\, \frac{\partial^2 C_t}{\partial S_t^2}\, \sigma_S^2 S_t^2 + \frac{1}{2}\, \frac{\partial^2 C_t}{\partial R_t^2}\, \sigma_R^2 + \frac{\partial^2 C_t}{\partial S_t \partial R_t}\, \sigma_S S_t \sigma_R \rho = R_t C_t, \qquad (14)$$

which, given (13), can be shortened to

$$\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t}\, R_t S_t + \frac{1}{2}\, \frac{\partial^2 C_t}{\partial S_t^2}\, \sigma_S^2 S_t^2 \quad = \quad R_t C_t, \qquad (15)$$

since the second derivative with respect to the interest rate as well as the mixed term vanishes. Again, this differential equation has to hold for any two values of R_t, $R_t^{(1)} \neq R_t^{(2)}$, and the pricing function C has to satisfy

$$\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t}\, R_t^{(i)} S_t + \frac{1}{2}\, \frac{\partial^2 C_t}{\partial S_t^2}\, \sigma_S S_t^2 \quad = \quad R_t^{(i)} C_t \qquad (16)$$

for $i = 1, 2$. Subtracting (16) for $R_t = R_t^{(2)}$ from the corresponding equation for $R_t = R_t^{(1)}$ and dividing by $R_t^{(1)} - R_t^{(2)}$ yields the following condition:

$$\frac{\partial C_t}{\partial S_t}\, S_t \quad = \quad C_t \qquad (17)$$

where we have used that C and therefore also its partial derivatives do not depend on R. This implies that the claim price is linear in the stock price

$$C_t \quad = \quad b(t) S_t, \qquad (18)$$

where $b(t)$ is at most a deterministic function of time. It has already been shown in the proof of Theorem 1 that $b(t)$ is even independent of t, i.e. $b(t) = b(0) \equiv b$. □

2.3 Discrete time model

In the previous subsection we have shown that it is impossible to generate path-independent claims that are not affine linear in S_T when we are working in a continuous-time model with stochastic volatility or stochastic interest rates. Discrete models are often used as powerful, but easy to understand analogues to continuous models, because due to the (usually) finite state space many of the mechanisms in the context of derivative pricing are much easier to handle technically. However, one should always make sure that a particular property of a continuous-time model actually carries over to the discrete case. The propositions derived in this paper are an example for a situation where the analogy between continuous and discrete models is not perfect. As we will see there are indeed attainable path-independent claims in discrete models with incomplete markets.

As an example for such a case consider a discrete version of a stochastic volatility model. The evolution of the stock price and the local volatility is shown in figure 1. The interest rate is set equal to zero for simplicity. The exact values of the local volatility at times $t = 0$ and $t = 1$ are not relevant, it is only important that there is a state where volatility is high (low) at time $t = 1$, so that $V_1 = V^u$ ($V_1 = V^d$).

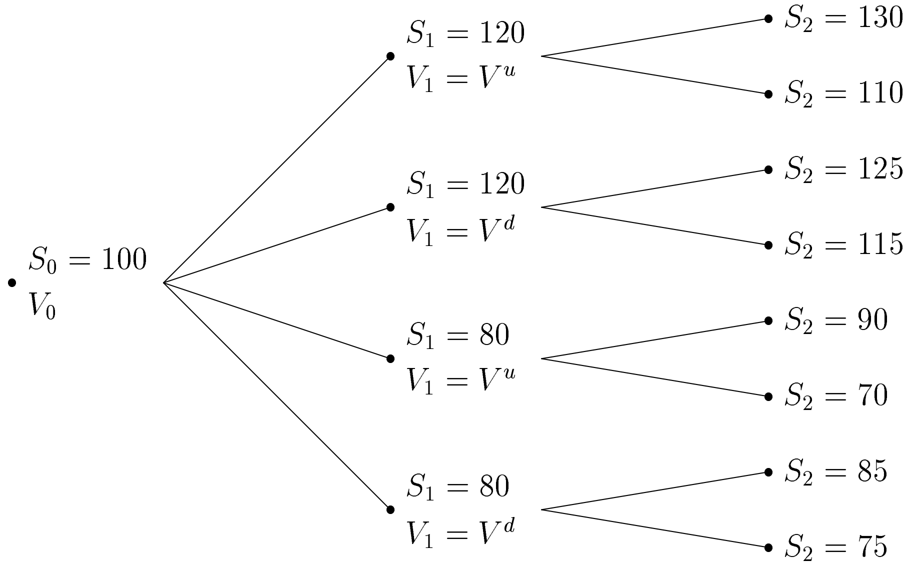

Figure 1: Stock price and volatility in the example

Consider now an arbitrary payoff $f(S_2)$. Clearly, over the second period, replication is always possible, since the value of V_2 does not matter for the determination of the payoff, resulting basically in a binomial model. In contrast to this replication over the first period is possible if and only if the value of the claim at time $t = 1$ only depends on the stock price, but not on volatility. This is a condition analogous to (3). To see that this restriction does not eliminate all nonlinear payoffs from the set of attainable time-T path-independent claims, consider a European call option with a strike price of 100 and maturity date $T = 2$. This claim is indeed attainable, since at time $t = 1$ one holds either one unit of the stock and 100 units of cash short (in the upper two states) or no stock and no cash at all (in the other two states). The respective portfolio values are 20 in the two upper states and zero in the two lower states. So at time $t = 0$ we would hold half a unit of the stock and 40 units of cash short with an initial cost of the self-financing strategy of 10.

A seemingly minor change in the contract specifications of the option, however, renders a completely different result. Instead of a strike of 100, now assume an exercise price of 120. Again, replication is possible over the second period. However, in contrast to the previous case this claim is not attainable, since the value at time $t = 1$ is different in the two states with $S_1 = 120$, i.e. there is an explicit dependence on V_1.

There is an important structural difference between the two options considered in the previous examples. If one was to draw a graph of the two payoff functions one would note that for the call with strike 100 the kink of the function is right between the upper and the lower subset of nodes at time $t = 2$. The call with strike 120 does not share this property, since the kink of its payoff function is between the two nodes following a stock price of 120 at $t = 1$ for both values of volatility at time $t = 1$.

3 Application: Bounds on Option Prices

We will now apply our main result to the problem of superhedging and subhedging. A superhedging (subhedging) strategy for a derivative asset is a trading strategy that yields a terminal value at time T which is greater (less) than or equal to the payoff of the claim with probability one. The value at time $t = 0$ of a superhedging (subhedging) portfolio is of course also an upper (lower) bound for the value of the claim itself. Recent papers dealing with superhedging and subhedging are, among others, Frey and Sin (1999), Frey (2000), Avellaneda, Levy, and Parás (1995), and Eberlein and Jacod (1997). Whereas Eberlein and Jacod (1997) analyze bounds on European option prices in an economy where asset prices follow Lévy processes, the other authors focus on situations where volatility is stochastic, but there also exist positive and finite bounds v_{max} and v_{min} such that $v_{min} \leq v_t \leq v_{max}$ with probability one.

In terms of superhedging, imposing volatility bounds of this type is basically equivalent to considering an 'artificial' BS economy. Since call prices are monotonic with respect to volatility, we can create a superhedge in the original economy by using the maximum volatility v_{max} in a BS hedge. This implies that the strategy is either exactly self-financing over the next interval, or money can be extracted from the hedge portfolio, but there is never any need for additional funds. It is exactly here, where the difference to a model with unbounded volatility becomes important, since with unbounded volatility only static buy-and-hold strategies will be self-financing or even allow the withdrawal of funds.

This means that in an SV model with unbounded volatility, searching for a self-financing superhedging portfolio for a European call actually means to search for a superhedging payoff that can be generated by a buy-and-hold strategy, which has to be affine linear in S_T. The optimization problem to be solved is therefore given by

$$\begin{aligned} &\min\nolimits_{a,b}\{a + bS_0\} \\ \text{s.t.} \quad & ae^{rT} + bS_T \geq (S_T - K)^+, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\min\nolimits_{a,b}\{a + bS_0\} \\ \text{s.t.} \quad & ae^{rT} + bS_T \geq \begin{cases} 0 & S_T \leq K \\ S_T - K & S_T > K \end{cases} \end{aligned}$$

The solution is given by $a^\star = 0, b^\star = 1$ so that $a^\star + b^\star S_0 = S_0$. To see this, let first S_T approach zero, yielding $a \geq 0$. On the other hand, for $S_T \geq K$, the restriction on the portfolio value is equivalent to $ae^{rT} + K + (b-1)S_T \geq 0$. Since this has to hold for arbitrarily large S_T, it must be true that $b - 1 \geq 0$ or $b \geq 1$. The objective function is then obviously minimized for $a^\star = 0$ and $b^\star = 1$. Therefore, in an SV model or an SVJ model, the cheapest superhedge for the call is the stock itself when volatility can take on any positive value.

For the problem of subhedging a European call we can apply the same technique as above. We want to find the most expensive linear payoff which is lower than or equal to the call payoff. The optimization problem then becomes

$$\begin{aligned} &\max_{a,b}\{a + bS_0\} \\ \text{s.t.} \quad & ae^{rT} + bS_T \leq (S_T - K)^+, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\max_{a,b}\{a + bS_0\} \\ \text{s.t.} \quad & ae^{rT} + bS_T \leq \begin{cases} 0 & S_T \leq K \\ S_T - K & S_T > K \end{cases} \end{aligned}$$

We know that $a^\star + b^\star S_0 = \max\{S_0 - Ke^{-rT}, 0\}$ so that for $S_0 < Ke^{-rT}$ both a^* and b^* will be equal to zero, whereas for $S_0 > Ke^{-rT}$ we obtain $a^* = -Ke^{-rT}$ and $b^\star = 1$. The proof works along the same lines as the one shown above for the superhedge. The interpretation is that in the models considered here, the subhedge for the call is either to do nothing or buy the stock and invest the discounted strike into the money market account, whichever is more expensive.

Finally, consider the problem of superhedging European options where the payoff is raised to some power greater than one, i.e. $g(S_T) = \left[(S_T - K)^+\right]^\gamma$ for some $\gamma > 1$. Since the slope of the payoff becomes steeper and steeper as S_T gets large, there is no affine linear function of S_T which could dominate the payoff of such an option. In other words, superhedging is not possible for this type of claims.

4 Conclusion

In this paper we have analyzed the problem of attainability of path-independent European claims via self-financing strategies in incomplete markets. Frequently discussed reasons for market incompleteness in the context of continuous-time derivative pricing are stochastic volatility and stochastic short rates. For the stochastic volatility model we have proved that the only attainable path-independent claims are affine linear in the stock with a simple buy-and-hold replication. This result may seem counterintuitive at a first glance, since one is tempted to assume that due to the rich class of possible dynamic trading strategies the set of attainable path-independent payoffs should be larger. The key point is that the strategies have to be self-financing which restricts their values to be independent of any other risk factors than the stock. Especially the sensitivities of the claim price with respect to volatility have to equal zero. For SV models this leads to a considerable simplification of the partial differential equation for the claim price, and finally allows us to deduce that the claim has to be linear in the stock price. When considering a model with stochastic interest rates, an analogous result can be proved along similar lines. Here the class of traded path-independent payoffs is even smaller, since only linear claims can be replicated.

Our findings have important implications for the determination of bounds on the prices of non-redundant claims when volatility is unbounded. For example, our result offers a simple way to deduce that the only superhedge for a call in an incomplete market is the stock itself. Furthermore, for derivatives with payoff functions representing a power greater than one in the stock price, there is no superhedge at all.

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