Fourier Inversion Algorithms for generalized CreditRisk$^+$ Models and an Extension to Incorporate Market Risk [1]

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Abstract

A popular model to describe credit risk in practice is CreditRisk$^+$ and in this paper a Fourier inversion to obtain the distribution of the credit loss is proposed. Therefore the CreditRisk$^+$ model is described in terms of characteristic functions. One advantage of this approach is that one can abstain from the basic loss unit, which was introduced in the CreditRisk$^+$ model for computational reasons only. To determine the credit loss distribution, two methods based one the Fourier transformation are established, which work even if the corresponding characteristic function is not integrable.

The CreditRisk$^+$ model will be extended such, that general dependent sector variables can be taken into consideration, for example dependent lognormal sector variables. The computation of the loss distribution for such generalized models is based on a combination of Monte Carlo simulation and Fourier inversion. The transfer to a continuous time model will be performed and the sector variables become processes, more precisely geometric Brownian motions.

To have a time continuous credit risk model is an important step to combine this model with market risk. Additionally a portfolio model will be presented where the changes of the spreads are driven by the sector variables. Using a linear expansion of the market risk, the distribution of this portfolio can be determined. In the special case that there is no credit risk, this model yields the well known Delta normal approach for market risk, hence a link between credit risk and market risk has been established.

1 Introduction

A quite popular way to describe credit risk is given by the CreditRisk$^+$ model presented by Credit Suisse First Boston in 1997. This is a two state model which is quite natural from point of view of a buy–and–hold investor who is only interested in the states *default* and *no default*. The CreditRisk$^+$ model is easy to implement and the technique of Panjer recursion is known from insurance risk models. But these recursion may be unstable for the determination of the Value at Risk and it is useful to have alternative algorithms like saddlepoint approximations presented in [10] or an algorithm based on Fourier inversion presented in this paper. There are additional shortcomings in the CreditRisk$^+$ approach; so there is the need in further development of credit risk models.

One disadvantage of the CreditRisk$^+$ model is the introduction of a basic loss unit and that all losses in the case of default must be integer multiple of this unit. Since a loss given default can range for a bank from a few hundred Euro (e.g. a credit card expose) up to more than a billion Euro (e.g. a large company loan), it is quite difficult to find a proper value for the loss unit. If it is chosen to be small, the computations take too much time and if it is taken too big, the results could have too big errors. Therefore the CreditRisk$^+$ model without assuming a basic loss unit is presented in section 2.

The result of the analysis is the characteristic function of the credit portfolio loss and its distribution will be determined by Fourier inversion. However, a plain Fourier inversion is not possible since the characteristic function is not integrable. The properties of the Fourier transformation are recalled and two methods to perform a Fourier inversion are presented, which apply to the CreditRisk$^+$ model. One method

is based on the structure of the Fast Fourier Transform algorithm and the second one uses an approximation based on the first two moments of the unknown distribution. Another disadvantage of the CreditRisk$^+$ model is the assumption of independent sector variables. A first study in [2] showed that there is an effect of dependent sector variables on the variance of the credit loss. Another generalization of the CreditRisk$^+$ model to dependent sector variables is presented in section 4. Since it is difficult in general to describe the dependency of random variables completely, it is suggested for the application in practice to introduce dependent lognormal sector variables, because the dependency of lognormal random variables can easily be described by a covariance matrix. The resulting model is more complex and the valuation can be done by Monte Carlo only, but the computational effort is tolerable.
The CreditRisk$^+$ model gives an answer to the question of the size of the losses *at the end* of a fixed period, but one would like to analyse the process of the losses *during* this period. Therefore a model with lognormal sector processes is introduced in section 5. The description of the default events in this model arise quite natural according to the CreditRisk$^+$ model.
With a model which allows time continuous description of credit risk one has made a large step to combine market and credit risk. In section 6 a portfolio valuation of a combined model is introduced, where the credit spreads are modelled by the sector processes. The profit and loss distribution of the portfolio can be obtained with the same techniques presented in the previous sections. One can consider two special cases of this model. First, if there is no market risk one obtains the model presented in section 5. In the special case, that the portfolio contains no credit risk, one gets the well known Delta normal approach to determine market risk.

2 The CreditRisk$^+$ Model

In this section a credit model is presented, which is very close to the CreditRisk$^+$ model [4]. The main difference is, that no basic loss unit is introduced, which is no longer necessary if one applies Fourier inversion techniques. Also a specific risk sector, the so–called "idiosyncratic risk", is introduced as proposed in the CreditRisk$^+$ manual [4][A 12.3].

2.1 Introduction of the Model

The aim of the following analysis is to characterize the losses which may occur in a loan portfolio with N obligors. In order to describe the reliability of the j^{th} obligor, a rating parameter p^j is introduced. As it will be shown below in lemma 1, p^j is the probability that the obligor j will default within one year.
In order to describe dependencies between the obligors K sectors are introduced. For each obligor the sector affiliations a_k^j and the idiosyncratic risk a_0^j are given and the following relations hold:

$$a_k^j \geq 0 \qquad \forall\, j = 1, \ldots, N\ ;\ k = 0, \ldots K \tag{1}$$

$$\sum_{k=0}^{K} a_k^j = 1 \qquad \forall\, j = 1, \ldots, N \tag{2}$$

The sectors in this model are described by independent Gamma distributed random variables R^k with expectation 1 and variance σ_k^2. The parameter $\sigma_k > 0$ is the volatility parameter of the kth sector. For each obligor a default intensity is now defined by

$$\lambda^j \quad := \quad p^j \left(a_0^j + \sum_{k=1}^{K} a_k^j R^k \right) =: p^j \tilde{\lambda}^j \tag{3}$$

Remark 1 *By definition, the following statements hold:*

$$\tilde{\lambda}^j \quad > \quad 0 \tag{4}$$

$$\mathbf{E}[\tilde{\lambda}^j] \quad = \quad 1 \tag{5}$$

Since one is interested in credit risk one may assume that $p^j > 0$ which is equivalent with the assumption, that it is possible for each obligor to default, hence $\lambda^j > 0$ [2]. Let Y^j be independent exponential distributed random numbers with intensity λ^j. Then interpret Y^j as the date of default of the j^{th} obligor. Let us concentrate on a fixed time horizon T and the jth obligor defaults, if $Y^j \leq T$. For simplicity in the following calculations the following binary random numbers are introduced:

$$I^j = \begin{cases} 1 & \text{if } Y^j \leq T \\ 0 & \text{else} \end{cases} \tag{6}$$

Let L^j denotes the loss if the obligor j defaults. So L^j is the so called "loss given default", that is the credit nominal times a non–random recovery rate for the investor. Hence the overall loss of the credit portfolio up to the fixed time horizon T is given by the random variable

$$X \quad := \quad \sum_{j=1}^{N} I^j L^j \tag{7}$$

Remark 2 *For a bank, typical values are $N \approx 10,000$ or even bigger, $K \lesssim 100$ and the losses given default L^j can range from a few hundred Euro for credit card exposure up to more than a billion Euro for large company loans. A typical value for the time horizon T in this context is 1 year.*

Assumption 1 *In the whole paper it is assumed, that the rating parameters p^j are small. By the next lemma it is equivalent to assume, that there is a low probability for each obligor to default within one year.*

Lemma 1 *For small p^j, the one year default probability is given by p^j:*

$$P[Y^j \leq 1] \quad = \quad p^j + \mathcal{O}\left((p^j)^2\right) \tag{8}$$

[2]This assumption is only necessary to define exponential distributed random numbers with intensity λ^j. But if one defines $I^j = 0$ almost sure in the case that $p^j = 0$, all following computations go through.

Proof. The default probability of obligor j conditioned on the state of the sector variables R is given by:

$$P[Y^j \leq 1|R] \;=\; 1 - e^{-\lambda^j} = 1 - e^{-p^j \tilde{\lambda}^j} \tag{9}$$

One can take the expectation over R and expand the exponential function:

$$\begin{aligned} P[Y^j \leq 1] &= \mathbf{E}[P[Y^j \leq 1|R]] = \mathbf{E}[1 - e^{-p^j \tilde{\lambda}^j}] && (10)\\ &= p^j \mathbf{E}[\tilde{\lambda}^j] + \mathcal{O}\left((p^j)^2\right) = p^j + \mathcal{O}\left((p^j)^2\right) && (11) \end{aligned}$$

■

2.2 The characteristic function of the loss distribution

The characteristic function of X conditioned on R is given by:

$$\Phi_{X|R}(s) \;=\; \prod_{j=1}^{N} \Phi_{L^j I^j|R}(s) = \prod_{j=1}^{N} \Phi_{I^j|R}(L^j s) \tag{12}$$

For the characteristic function of I^j conditioned on R one obtains:

$$\begin{aligned} \Phi_{I^j|R}(s) &= \mathbf{E}[e^{isI^j}|R] = \int_0^T e^{is1}\lambda_j e^{-\lambda^j t} dt + \int_T^\infty e^{is0}\lambda^j e^{-\lambda^j t} dt && (13)\\ &= e^{is}(1 - e^{-\lambda^j T}) + e^{-\lambda^j T} && (14) \end{aligned}$$

Lemma 2 *For all $x \geq 0$ and $s \in \mathbb{R}$ holds:*

$$\left|\left(e^{is}(1 - e^{-x}) + e^{-x}\right) - \left(e^{-x(1-e^{is})}\right)\right| \;\leq\; 3x^2 \tag{15}$$

Proof. Fix $s \in \mathbb{R}$ and define the function

$$f_s(x) \;:=\; e^{is}(1 - e^{-x}) + e^{-x} - e^{-x(1-e^{is})} \tag{16}$$

$f_s(x)$ is two times continuous differentiable; the first two derivatives are given by:

$$\begin{aligned} f_s'(x) &= e^{is}e^{-x} - e^{-x} + (1 - e^{is})e^{-x(1-e^{is})} && (17)\\ f_s''(x) &= -e^{is}e^{-x} + e^{-x} - (1 - e^{is})^2 e^{-x(1-e^{is})} && (18) \end{aligned}$$

Since $f_s(0) = 0$, $f_s'(0) = 0$ and $|f_s''(x)| \leq 6$ for all $x \geq 0$, one obtains the statement of the lemma by the Taylor theorem. ■

Since $\lambda^j T$ is proportional to p^j and hence small, one may approximate by lemma 2:

$$\Phi_{I^j|R}(s) \;\approx\; \exp\left(\lambda^j T(e^{is} - 1)\right) \tag{19}$$

This is the characteristic function of the Poisson distribution. Let $f^j(s)$ denote the error of this approximation, then this error is bounded by $|f^j(s)| \leq 3(\lambda^j T)^2$. One thus obtains together with (3) and (12):

$$\Phi_{X|R}(s) \;=\; \exp\left(\sum_{j=1}^{N} p^j \left(a_0^j + \sum_{k=1}^{K} a_k^j R^k\right) T(e^{iL^j s} - 1)\right) + F(s) \tag{20}$$

The error term $F(s)$ comes from the products of the characteristic functions $\Phi_{I^j|R}(s)$, each with an bounded error $f^j(s)$. Since characteristic functions are bounded by 1, the overall error $F(s)$ of the conditioned characteristic function $\Phi_{X|R}(s)$ is bounded by powers of $(\lambda^j T)^2$. The conditioning on R will be solved by taking the expectation and terms of order $\mathcal{O}((p^j)^2)$ will be neglected[3]. Recall, that R^k are independent gamma distributed with mean 1 and variance σ_k^2, hence:

$$\begin{aligned}
\Phi_X(s) &= \mathbf{E}\left[\Phi_{X|R}(s)\right] && (21)\\
&= \mathbf{E}\left[\exp\left(\sum_{j=1}^{N} p^j \left(a_0^j + \sum_{k=1}^{K} a_k^j R^k\right) T(e^{iL^j s}-1)\right)\right] && (22)\\
&= e^{\sum_{j=1}^{N} a_0^j p^j T(e^{iL^j s}-1)}\, \mathbf{E}\left[\exp\left(\sum_{k=1}^{K} R^k \sum_{j=1}^{N} p^j a_k^j T(e^{iL^j s}-1)\right)\right] && (23)\\
&= e^{\sum_{j=1}^{N} a_0^j p^j T(e^{iL^j s}-1)} \prod_{k=1}^{K} \mathbf{E}\left[\exp\left(\sum_{j=1}^{N} p^j a_k^j T(e^{iL^j s}-1) R^k\right)\right] && (24)
\end{aligned}$$

By well known properties of the gamma distribution one obtains

$$\Phi_X(s) = \exp\left(\sum_{j=1}^{N} a_0^j p^j T(e^{iL^j s}-1)\right) \prod_{k=1}^{K} \left(\frac{1}{1+\sigma_k^2 \sum_{j=1}^{N} p^j a_k^j T(1-e^{iL^j s})}\right)^{\frac{1}{\sigma_k^2}} \qquad (25)$$

Using the main branch of the logarithm, one can rewrite the characteristic function:

$$\Phi_X(s) = \exp\left(\sum_{j=1}^{N} a_0^j p^j T(e^{iL^j s}-1) - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \ln\left[1+\sigma_k^2 T \sum_{j=1}^{N} a_k^j p^j (1-e^{iL^j s})\right]\right) \qquad (26)$$

This is the characteristic function of the losses of the credit portfolio and one aims to determine the distribution of X. But this characteristic function is not integrable and a usual Fourier inversion fails. First, some properties of X are studied, which can be obtained directly from the characteristic function. These properties are the first moments and the distribution in the limit of an infinite number of homogeneous obligors. In section 3 more sophisticated Fourier inversion techniques will be presented to determine the distribution of X.

2.3 The First Moments of X

The cumulant generating function $C_X(s)$ of a random variable X is defined by $C_X(s) := \ln \mathbf{E}[e^{sX}]$, if this expectation exists and then holds the general relationship

[3]The error analysis holds for any distribution of R as long as all moments of R exist. In the case of the Gamma distribution, one can show more easily, that the error is of order $\mathcal{O}((p^j)^2)$. However, for generalizations of the CreditRisk$^+$ model it is useful to have an error analysis which is independent of certain properties of the distribution.

$C(s) = \ln \Phi(-is)$ between the cumulant generation function and the characteristic function. Hence, the cumulant generating function of the credit loss is given by:

$$C_X(s) = \sum_{j=1}^{N} a_0^j p^j T(e^{L^j s} - 1) - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \ln \left[1 + \sigma_k^2 T \sum_{j=1}^{N} a_k^j p^j (1 - e^{L^j s}) \right] \quad (27)$$

The cumulant generating function can be used to prove the

Theorem 1 *The first two moments of X are given by:*

$$\mathbf{E}[X] = T \sum_{j=1}^{N} p^j L^j \quad (28)$$

$$\mathbf{Var}[X] = T \sum_{j=1}^{N} p^j (L^j)^2 + T^2 \sum_{k=1}^{K} \sigma_k^2 \left(\sum_{j=1}^{N} a_k^j p^j L^j \right)^2 \quad (29)$$

$$\mathbf{E}[X^2] = T \sum_{j=1}^{N} p^j (L^j)^2 + T^2 \sum_{k=1}^{K} \sigma_k^2 \left(\sum_{j=1}^{N} a_k^j p^j L^j \right)^2 + T^2 \left(\sum_{j=1}^{N} p^j L^j \right)^2 \quad (30)$$

Proof. Compute the first and second derivative of $C_X(s)$ to determine the first two cumulants of X:

$$C'(s) = T \sum_{j=1}^{N} a_0^j p^j e^{L^j s} L^j + \sum_{k=1}^{K} \frac{T \sum_{j=1}^{N} a_k^j p^j e^{L^j s} L^j}{1 + \sigma_k^2 T \sum_{j=1}^{N} a_k^j p^j (1 - e^{L^j s})} \quad (31)$$

$$C''(s) = T \sum_{j=1}^{N} a_0^j p^j e^{L^j s} (L^j)^2 + \sum_{k=1}^{K} \frac{T \sum_{j=1}^{N} a_k^j p^j e^{L^j s} (L^j)^2}{1 + \sigma_k^2 T \sum_{j=1}^{N} a_k^j p^j (1 - e^{L^j s})} + \sum_{k=1}^{K} \frac{\sigma_k^2 \left(T \sum_{j=1}^{N} a_k^j p^j e^{L^j s} L^j \right)^2}{\left(1 + \sigma_k^2 T \sum_{j=1}^{N} a_k^j p^j (1 - e^{L^j s}) \right)^2} \quad (32)$$

Note, that the first cumulant is the expectation and the second cumulant is the variance. So the valuation of these expressions at $s = 0$ yields:

$$\mathbf{E}[X] = T \sum_{j=1}^{N} a_0^j p^j L^j + T \sum_{j=1}^{N} \sum_{k=1}^{K} a_k^j p^j L^j = T \sum_{j=1}^{N} p^j L^j \quad (33)$$

$$\mathbf{Var}[X] = T \sum_{j=1}^{N} p^j (L^j)^2 + T^2 \sum_{k=1}^{K} \sigma_k^2 \left(\sum_{j=1}^{N} a_k^j p^j L^j \right)^2 \quad (34)$$

$$\mathbf{E}[X^2] = \mathbf{Var}[X] + \mathbf{E}[X]^2 \quad (35)$$

■

Using the same idea one can of course also compute the next cumulants, but their explicit knowledge is not so important for our analysis. The introduction of the idiosyncratic risk does not effect the mean of X, but there is an influence on the variance of the resulting distribution; it reduces the variance.

2.4 The infinite large and homogeneous loan portfolio

Since the number of obligors N is quite large in typical credit portfolios, one may think about the distribution of a portfolio with an infinite number of obligors. A special case of such an portfolio limit is presented here, more general results can be found in [3]. Even if this discussion seems to be rather theoretical than practical, one gets a good impression, how the distribution of the losses of a large loan portfolio will look like.
Define a sequence of credit portfolios with losses X_n and assume that all loans are influenced by one sector only. The idea is to consider portfolios with an infinite number of loans, where each loan is of infinitesimal size:

Definition 1 *Define a series of loan portfolios, using the previous notations, by*

$$
\begin{aligned}
K^{(n)} &= 1 && (36)\\
\sigma_1^{(n)} &> 0 && (37)\\
a_0^{j(n)} &= a_0^{(n)} && (38)\\
a_1^{j(n)} &= a_1^{(n)} = 1 - a_0^{(n)} && (39)\\
L^{j(n)} &= L^{(n)} && (40)\\
p^{j(n)} &= p^{(n)} && (41)
\end{aligned}
$$

Let further hold:

$$
\begin{aligned}
\lim_{n\to\infty} a_0^{(n)} &= a_0 > 0 && (42)\\
\lim_{n\to\infty} L^{(n)} &= 0 && (43)\\
\lim_{n\to\infty} N^{(n)} &= \infty && (44)\\
\lim_{n\to\infty} N^{(n)} p^{(n)} L^{(n)} &= c \qquad \textit{with } 0 < c < \infty && (45)
\end{aligned}
$$

Then the series $X^{(n)}$, defined as the loss of the nth *portfolio, is a series of homogeneous loan portfolios.*

Theorem 2 *Let $X^{(n)}$ be a series of homogeneous loan portfolios. Then the limit $X^\infty := \lim_{n\to\infty} X^{(n)}$ exists and the distribution of X^∞ is given by a shifted Gamma distribution.*

Proof. Using the definition of a series of homogeneous loan portfolios, one can write down the characteristic function of $X^{(n)}$ for any fixed s using (25):

$$
\Phi_{X^{(n)}}(s) = e^{\sum_{j=1}^{N^{(n)}} a_0^{(n)} p^{(n)} T(e^{iL^{(n)}s}-1)} \left(\frac{1}{1+\sigma_1^2 \sum_{j=1}^{N^{(n)}} p^{(n)} a_1^{(n)} T(1-e^{iL^{(n)}s})} \right)^{\frac{1}{\sigma_1^2}} \qquad (46)
$$

$$= e^{N^{(n)} a_0^{(n)} p^{(n)} T i L^{(n)} s} \left(\frac{1}{1 - i\sigma_1^2 N^{(n)} a_1^{(n)} p^{(n)} T L^{(n)} s} \right)^{\frac{1}{\sigma_1^2}} + o(1) \qquad (47)$$

Taking the limit for $n \to \infty$:

$$\Phi_{X^\infty}(s) = \exp(i a_0 c T s) \left(\frac{1}{1 - i\sigma_1^2 a_1 T c s} \right)^{\frac{1}{\sigma_1^2}} \qquad (48)$$

This is the characteristic function of the shifted gamma distribution. ■

3 Fourier Inversion Techniques for CreditRisk$^+$

From the last section, some properties and the characteristic function of the portfolio loss X are known. The aim is now to determine the distribution X. In many cases, a Fourier inversion of the characteristic function will give the density. But if $\Phi_X(s)$ is not integrable – that is the situation one is faced with in the CreditRisk$^+$ model, the straight Fourier inversion integral does not exist. Nevertheless one can obtain the distribution of X by Fourier inversion techniques.

First, some properties of the Fourier transformation are recalled. Then two methods are presented, which can be applied in the context of CreditRisk$^+$, i.e. which can be used to determine the distribution of X. The first method is based on the very special structure of the well known Fast Fourier Transformation (FFT) algorithm. The idea of the second method is to approximate the unknown distribution by another distribution with the same mean and variance. Then the Fourier inversion can be used to determine the difference between the two distributions.

3.1 The Fourier Transformation

Some known facts about the Fourier transformation and characteristic functions are collected. The proofs of these results can be found in several textbooks, e.g. [8].

Definition 2 *A function $f : \mathbb{R} \to \mathbb{R}$ is a L^1 function, if*

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty \qquad (49)$$

Definition 3 *Let f be L^1. Then the Fourier transform of f exists and is defined by*

$$\Phi_f(s) := \int_{-\infty}^{\infty} e^{isx} f(x) dx \qquad (50)$$

Remark 3 *Let f be a probability density, that is $f(x) \geq 0$ and $\int f(x)dx = 1$. Then the Fourier transform of f exists and $\Phi_f(s)$ is called the characteristic function. Even in the case, that a real–valued random variable X has no density, the characteristic function defined by $\mathbf{E}[e^{isX}]$ always exist.*

Theorem 3 (Fourier Inversion) *Let Φ_f be the characteristic function of a distribution F and let Φ_f be L^1. Then F has a continuous density f given by:*

$$f(x) \quad = \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \Phi_f(s) ds \tag{51}$$

Lemma 3 *Let f be a function and Φ_f its Fourier transform. Then the following inequalities hold whenever the integrals on the right hand side are defined:*

$$|\Phi_f(s)| \quad \leq \quad \int_{-\infty}^{\infty} |f(x)| dx \tag{52}$$

$$|f(x)| \quad \leq \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_f(s)| ds \tag{53}$$

Proof. The first equation is a simple inequality:

$$|\Phi_f(s)| \quad = \quad \left| \int_{-\infty}^{\infty} e^{isx} f(x) dx \right| \leq \int_{-\infty}^{\infty} |e^{isx}||f(x)| dx = \int_{-\infty}^{\infty} |f(x)| dx \tag{54}$$

The second equation follows similarly from the Fourier inversion theorem. ∎

Remark 4 *If Φ is a characteristic function, then Φ is bounded by* 1.

Lemma 4 *Let f be L^1 and Φ_f its Fourier transform. Then holds*

$$\lim_{s \to \pm\infty} \Phi_f(s) = 0 \tag{55}$$

If f has an nth *derivative $f^{(n)}$ and if $f^{(n)}$ is L^1, then*

$$\lim_{s \to \pm\infty} |s|^n \Phi_f(s) = 0 \tag{56}$$

Remark 5 *If f is a two times differentiable density and f'' is L^1, then the characteristic function Φ_f is also integrable.*

Lemma 5 *Let F be an arbitrary distribution function and Φ_f its corresponding characteristic function. Define the absolute moments by*

$$M_n \quad := \quad \int_{-\infty}^{\infty} |x|^n dF(x) \tag{57}$$

If $M_n < \infty$ then the nth *derivative of Φ_f exists and is given by*

$$\Phi_f^{(n)}(s) \quad = \quad i^n \int_{-\infty}^{\infty} e^{isx} x^n dF(x) \tag{58}$$

Remark 6 *The moments of a distribution determine the distribution uniquely, if*

$$\lambda \;:=\; \limsup_{k\to\infty} \sqrt[k]{\frac{M_k}{k!}} = e \limsup_{k\to\infty} \frac{1}{k} \sqrt[k]{M_k} < \infty \tag{59}$$

and $\frac{1}{\lambda}$ *is the convergence radius of the characteristic function* $\sum_{k=0}^{\infty} \frac{1}{k!} m_k (is)^k$, *where* m_k *denotes the* kth *moment. That is a sufficient condition; sufficient and necessary is the Carleman's condition*

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt[2k]{M_{2k}}} = \infty \tag{60}$$

3.2 The FFT based Fourier Inversion

It is obvious from the definition of X that the distribution of X has no density, but for a large number of obligors one would expect that the distribution is "almost" continuous. So one may look for an approximative density f of the random variable X.

For the approximation f is now required, that the following condition holds for each g in a suitable class of functions $\mathcal{G}$:

$$\int f(x)g(x)dx \;=\; \mathbf{E}[g(X)] \tag{61}$$

and a proper choice of $\mathcal{G}$ is given by

$$\mathcal{G} \;:=\; \{g(x) = e^{isx} | s \in \mathcal{S}\} \tag{62}$$

where $\mathcal{S}$ is a finite set of real numbers.

Note, that the functions $g \in \mathcal{G}$ are linear independent, which is necessary to expect a suitable approximation. The expectation in (61) for these functions g is the characteristic function evaluated at some points s with $s \in \mathcal{S}$. Since the characteristic function is known, the density f is the only unknown in equation (61) and hence one can use this equation to determine f. It is natural to expect that f will be a good approximation.

Now we are at the point, where the structure of the FFT algorithm will be used. Let us assume, that the unknown density f vanishes outside a given interval $[a, b]$[4]. Then its Fourier transform can be numerically computed by the following algorithm. Let S be the number of sample points (usually a power of 2) and define $\Delta x := \frac{b-a}{S-1}$ and $\Delta s := \frac{2\pi}{S\Delta x}$. Further define three S dimensional vectors and a $S \times S$ matrix M for $j, k = 0, \dots S-1$ by:

$$x_k \;:=\; a + k\Delta x \tag{63}$$

$$f_k \;:=\; f(x_k) \tag{64}$$

$$s_k \;:=\; \begin{cases} k\Delta s & \text{if } k < \frac{S}{2} \\ (k-S)\Delta s & \text{else} \end{cases} \tag{65}$$

$$M_{jk} \;:=\; \exp\left(2\pi i \frac{jk}{S}\right) \tag{66}$$

[4]In application to CreditRisk$^+$ this interval is naturally given by $a = 0$ (the loss if no obligor defaults) and $b = \sum_{j=1}^{N} L^j$ (the loss if all obligors default).

The set $\mathcal{S}$ is given by $\mathcal{S} := \{s_k | k = 0, \dots S-1\}$ and the Fourier transform of f at the points s_k will be denoted by $\Phi_k := \Phi_X(s_k)$.
The Fourier integral can be computed by the approximation

$$\int_{-\infty}^{\infty} e^{isx} f(x) dx \approx \Delta x e^{isa} \sum_{k=0}^{S-1} e^{isk\Delta x} f(a + k\Delta x) \qquad (67)$$

Using the vector and matrix notations, this equation yields[5]:

$$\Phi_k = \Delta x e^{ias_k} \sum_{j=0}^{S-1} M_{kj} f_j \qquad (68)$$

Having chosen a suitable interval $[a, b]$ and the number of Fourier steps S, the density f_k is the only unknown in equation (68). To determine f_k one has to solve the system of linear equations, which can be solved easily, since the inverse of M_{jk} is given by:

$$M_{jk}^{-1} = \frac{1}{S} \exp\left(-2\pi i \frac{jk}{S}\right) \qquad (69)$$

A short calculation shows, that this is in fact the inverse:

$$\sum_{k=0}^{S-1} M_{jk} M_{kl}^{-1} = \frac{1}{S} \sum_{k=0}^{S-1} \left(e^{2\pi i \frac{j-l}{S}}\right)^k = \begin{cases} 1 & j = l \\ \frac{1}{S} \frac{1-e^{2\pi i (j-l)}}{1-e^{2\pi i (j-l)/S}} = 0 & j \neq l \end{cases} \qquad (70)$$

Hence the inversion formula based on the linear equations is given by

$$f_j = \frac{1}{S\Delta x} \sum_{k=0}^{S-1} \exp\left(-2\pi i \frac{jk}{S}\right) e^{-ias_k} \Phi_k \qquad (71)$$

Since M^{-1} has a quite similar structure as M, one can compute the matrix vector multiplication by the FFT algorithm to save computation time. So even if the characteristic function is not integrable, the Fourier inversion by FFT works, since the algorithm does not compute an integral, but it solves a set of linear equations which describe the Fourier transformation of the unknown density f.

3.2.1 Smoothing the FFT Result

In the most cases, one expects that the density f is a smooth function, but the result of the FFT algorithm may have a sawtooth pattern. This pattern is in fact a perturbation with the so–called Nyquist critical frequency $\frac{S}{2}\Delta s = \frac{\pi}{\Delta x}$, which corresponds with the cut–off of the domain of the characteristic function. To eliminate this perturbation, one can apply the following algorithm, which performs a midpoint interpolation and so the number of sample points will be reduced by 1:

[5]Using the special form of M_{jk} one can valuate the matrix vector product by $\mathcal{O}(S \ln S)$ instead of the usual $\mathcal{O}(S^2)$ operations, which is the reason why this kind of Fourier transformation is called "Fast".

1. Smoothing algorithm: Elimination of a periodic perturbation
Let f be a density sampled on equidistant points x_k, $k = 0, \ldots, S-1$. Define

$$\tilde{f}_k := \frac{1}{2}(f(x_k) + f(x_{k-1})) \qquad k = 1, \ldots, S-1 \tag{72}$$

Then $\tilde{f}$ is a smooth version of the density f sampled on the points $x_k - \frac{1}{2}\Delta x$, $k = 1, \ldots, S-1$.

Another smoothing technique is needed at some positions, where the density tends to infinity which can happen in two situations. Either the density may not exist at a certain point x because $P[X = x] > 0$, hence the density is a Dirac δ function at x. Or the density tends to infinity at x although $P[X = x] = 0$, an example for such an density gives the χ^2 distribution with one degree of freedom. Then the FFT result may be very oscillatory near x and for some k one obtains $f_k = 0$. But a density must be non-negative! In such situations one the next following trick will help:

2. Smoothing algorithm: Smoothing near singularities
Let f be a density sampled on equidistant points x_k, $k = 0, \ldots, S-1$. If there are inner points such that the density is negative, hence $f_k < 0$ for a k in $1, \ldots, S-2$, then perform the following algorithm for each such k:

$$\begin{aligned}
left &:= (f_{k-1})^+ \\
right &:= (f_{k+1})^+ \\
f_{k-1} &:= f_{k-1} + \frac{left}{left + right} f_k \\
f_{k+1} &= f_{k+1} + \frac{right}{left + right} f_k \\
f_k &:= 0
\end{aligned}$$

For applications in practice, these smoothing methods which have both the computational effort of $\mathcal{O}(S)$ are rather fast and hence it is recommended to smooth a density by three steps:

1. Apply the first smoothing algorithm.
2. Perform the second smoothing algorithm on the result.
3. Finally apply the first smoothing algorithm again.

Since the application of the first smoothing algorithm shifts the x-domain by $\frac{1}{2}\Delta x$, this three step algorithm effects a shift of the domain by Δx.

3.3 Fourier Inversion using an Approximate Density

The idea of this method is not to obtain the density, but the integral of the distribution function. Even if the density is only defined in terms of Dirac δ functions, this function will be continuous and hence it is a good candidate for numerical computations. In order to use this method, one needs to know the first two moments of the unknown distribution, because the idea is to approximate the unknown distribution by a distribution with the same mean and variance. Then the Fourier inversion can be done using the following

Theorem 4 *Let F, G be the distribution functions of two distributions with existing third absolute moment, both with the same mean and variance, and let Φ_f, Φ_g denote their characteristic functions. Further define*

$$\hat{F}(x) := \int_{-\infty}^{x} F(y)\, dy, \qquad \hat{G}(x) := \int_{-\infty}^{x} G(y)\, dy \tag{73}$$

Then the following inversion formula holds:

$$\hat{F}(x) \;=\; \hat{G}(x) - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \frac{\Phi_f(s) - \Phi_g(s)}{s^2}\, ds \tag{74}$$

The theorem applies exactly for the situation one is faced with in the context of the CreditRisk$^+$ model. The characteristic function and the first two moments of an unknown density f are given and one can approximate f by another density g with the same first and second moment. For the application in the context of CreditRisk$^+$ a good choice for g is the Gamma distribution. Its characteristic function is explicitly known and the two parameters of this distribution can be expressed by the mean and variance. Additional, for large and homogeneous loan portfolios one expects a loss distribution which is almost gamma distributed.
Of course, one can also use other densities for the approximation. But since $X \geq 0$ by definition, one should concentrate on distributions on $\mathbb{R}^+$. So the lognormal distribution or the generalized gamma distribution (without shift and fitted to the first three moments) will also work, but the handling of these distributions is more expensive. The advantage in using such distribution could be, that the valuation of the Fourier integral becomes more accurate, or faster if a smaller number of sample points suffices.
As a result, one obtains the difference between $\hat{F}$ and $\hat{G}$. Since $\hat{G}$ is also known for the gamma distribution one only needs to differentiate $\hat{F}$ to obtain the distribution function of the loss variable X and the second differentiation yields the density f.
The advantage of this approach is, that one is independent of a special algorithm and so one can use more sophisticated integration methods than FFT for the valuation of the integral.

Remark 7 (Excursus on the Gamma distribution) *Let $\lambda, a > 0$. Then the density of the Gamma distribution is given by*

$$g(x) \;=\; \frac{\lambda}{\Gamma(a)} (\lambda x)^{a-1} e^{-\lambda x} \qquad x \geq 0 \tag{75}$$

Let X be Gamma distributed with λ and a. Then holds:

$$\mathbf{E}[X] \;=\; \frac{a}{\lambda} \tag{76}$$

$$\mathbf{Var}[X] \;=\; \frac{a}{\lambda^2} \tag{77}$$

Additional, the following statements also hold for $x \geq 0$:

$$G(x) = P(a, \lambda x) \tag{78}$$

$$\hat{G}(x) = xP(a, \lambda x) - \frac{a}{\lambda} P(a+1, \lambda x) \tag{79}$$

$$\Phi_g(s) = \frac{\lambda^a}{(\lambda - is)^a} \tag{80}$$

where P denotes the incomplete gamma function (see e.g. [1, 11]).

3.3.1 Proof of Theorem 4

Lemma 6 *Let two distributions have an existing third absolute moment. Let also have the distributions the same first and the same second moment. Then holds for the characteristic functions Φ_f, Φ_g of these distributions:*

$$\lim_{s \to 0} \frac{\Phi_f(s) - \Phi_g(s)}{s^2} = 0 \tag{81}$$

Proof. Since the third moment exists, the characteristic functions may be expanded near 0:

$$\Phi_f(s) = 1 + i\mu_1 s + \frac{i^2}{2}\mu_2 s^2 + \mathcal{O}(s^3) \tag{82}$$

$$\Phi_g(s) = 1 + i\mu_1 s + \frac{i^2}{2}\mu_2 s^2 + \mathcal{O}(s^3) \tag{83}$$

Hence $\Phi_f(s) - \Phi_g(s) = \mathcal{O}(s^3)$, what proves the lemma. ■

Lemma 7 *Let F and G be two distribution functions of two distributions with the same mean μ and let $\hat{F}$ and $\hat{G}$ be their integrals. Then holds:*

$$\lim_{x \to \pm\infty} (\hat{F} - \hat{G}) = 0 \tag{84}$$

Proof.
From the existence of the expectation one can conclude:

$$\int_x^\infty y \, dF(y) = o(1) \qquad \text{as } x \to \infty \tag{85}$$

Since $o(1) = \int_x^\infty y \, dF(y) \geq x \int_x^\infty dF(y) = x(1 - F(x))$ holds:

$$F(x) = 1 - o(\frac{1}{x}) \qquad \text{as } x \to \infty \tag{86}$$

The same asymptotic behaviour also holds for G. Since both distributions have the same mean, holds:

$$o(1) = \lim_{x \to \infty} \int_{-\infty}^{x} y \, d(F - G)(y) \tag{87}$$

$$= \lim_{x\to\infty} [y(F(y)-G(y))]_{-\infty}^{x} - \int_{-\infty}^{x} F(y)-G(y)\ dy \quad (88)$$

$$= \lim_{x\to\infty} x(F(x)-G(x)) - (\hat{F}(x)-\hat{G}(x)) \quad (89)$$

$$= \lim_{x\to\infty} \hat{G}(x)-\hat{F}(x) \quad (90)$$

The case $x \to -\infty$ is trivial, since $\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} G(x) = 0$. ■

Proof of Theorem 4. Let F, G fulfill the conditions of the theorem. By definition of the characteristic function holds:

$$\Phi_f(s) - \Phi_g(s) = \int_{-\infty}^{\infty} e^{isx}\ d(F-G)(x) \quad (91)$$

$$= \left[(F(x)-G(x))e^{isx}\right]_{-\infty}^{\infty} - is\int_{-\infty}^{\infty}(F(x)-G(x))e^{isx}dx \quad (92)$$

Since $\lim_{x\to\pm\infty} F(x)-G(x) = 0$ and e^{isx} is bounded, one obtains:

$$\Phi_f(s) - \Phi_g(s) = -is\int_{-\infty}^{\infty}(F(x)-G(x))e^{isx}dx \quad (93)$$

Lemma 7 and another integration by parts yield:

$$\Phi_f(s) - \Phi_g(s) = -is\left[\left(\hat{F}(x)-\hat{G}(x)\right)e^{isx}\right]_{-\infty}^{\infty} + (is)^2\int_{-\infty}^{\infty}\left(\hat{F}(x)-\hat{G}(x)\right)e^{isx}dx \quad (94)$$

$$\frac{\Phi_f(s) - \Phi_g(s)}{(is)^2} = \int_{-\infty}^{\infty}\left(\hat{F}(x)-\hat{G}(x)\right)e^{isx}dx \quad (95)$$

So $\frac{\Phi_f(s)-\Phi_g(s)}{(is)^2}$ is the Fourier transform of $\hat{F}(x)-\hat{G}(x)$. The left hand side of the last equation is well defined (see lemma 6) and is integrable since it decays with $\frac{1}{s^2}$ because Φ_f and Φ_g are bounded by 1. Therefore one may apply the Fourier inversion formula (51) to obtain:

$$\hat{F}(x)-\hat{G}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Phi_f(s)-\Phi_g(s)}{(is)^2}e^{-isx}ds \quad (96)$$

■

4 Lognormal or other Sector Variables

In the CreditRisk$^+$ model, the sector variables are assumed to be independent. From a practitioners point of view, this assumption is too rigorous, since one would like

to introduce the sectors due to general classes of business and typical sectors could be *construction*, *banking*, *utility industry*, *transportation*, etc. Also economic sectors like gross domestic product or the business activity of certain countries or currency areas can be taken into account. Using such sectors, it is quite easy to estimate the sector affiliations of each obligor, but these sectors are obviously not independent. One way to abstain from the independence assumption is presented in [2]; another is the topic of this section.
In the CreditRisk$^+$ model the sector variables R^k are assumed to be independent gamma distributed. The reason for this assumption is based on the simple calculations resulting from this supposition and not on statistical evidence. Hence one may suggest another distribution for the sector variables and one suggestion is to use dependent lognormal sectors. The advantage of this choice is, that the dependence of the sectors can be easily described by a correlation matrix. To estimate these correlations one can use as a first approximation the correlations between the corresponding asset indices. On the other hand, the characteristic function of the credit loss can not be obtained by a closed form solution any more, but it can be computed numerically by a Monte Carlo method.
To show, that the changeover from the Gamma distribution to the lognormal distribution has no dramatic impact on the distribution, the case of independent lognormal sector variables is shortly discussed. In this case, the difference between this model and the CreditRisk$^+$ model is rather small, because the gamma distribution and the lognormal distribution each with mean 1 and variance σ^2 look quite similar. It is shown in 4.2, that the expectation and the variance of the credit loss are also equal in this situation.

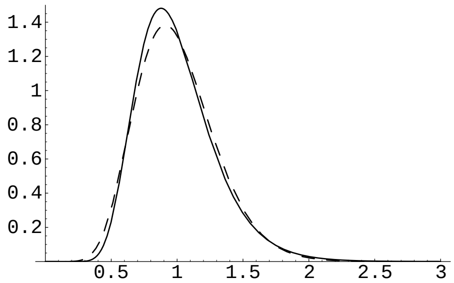

Figure 1: The lognormal density (solid) and the density of the gamma distribution (dashed); Both with mean 1 and $\sigma = 0.3$.

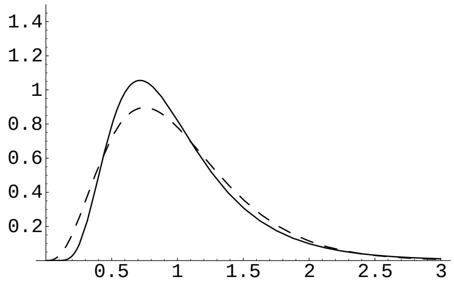

Figure 2: The lognormal density (solid) and the density of the gamma distribution (dashed); Both with mean 1 and $\sigma = 0.5$.

However, the computations in this section are valid for any jointly distributed sector variables R^k – as long as all pairwise covariances exist.

4.1 Introduction of the Model

This model is similar to the CreditRisk$^+$ model presented in section 2, but now the sector variables are dependent distributed, such that $\mathbf{Cov}[R^k, R^l]$ exists and $\mathbf{E}[R^k] = 1$ holds.
Since the other ingredients of the model are unchanged, one can inherit the charac-

teristic function of the credit losses conditioned on R, see (20):

$$\Phi_{X|R}(s) = \exp\left(\sum_{j=1}^{N} p^j T \left(a_0^j + \sum_{k=1}^{K} a_k^j R^k\right)(e^{iL^j s} - 1)\right) \tag{97}$$

In order to obtain the characteristic function of the credit loss X one has to take the expectation over R. One approach to solve this problem is to use a Monte Carlo method to compute the characteristic function of X. But before the Monte Carlo approach is studied, the mean and the variance of X under dependent sector variables will be determined. Since there is no closed form solution for the characteristic function $\Phi_X(s)$ one has to change the way to get the moments: Invert the order of differentiation and integration.

4.2 The first Moments of X

Theorem 5 *The first two moments of X are given by:*

$$\mathbf{E}[X] = T\sum_{j=1}^{N} p^j L^j \tag{98}$$

$$\mathbf{E}[X^2] = T\sum_{j=1}^{N} p^j (L^j)^2 + T^2\left[\sum_{j=1}^{N} p^j L^j\right]^2 + T^2 \sum_{j,n=1}^{N} p^j L^j p^n L^n \sum_{k,l=1}^{K} a_k^j a_l^n \mathbf{Cov}[R^k, R^l] \tag{99}$$

$$\mathbf{Var}[X] = T\sum_{j=1}^{N} p^j (L^j)^2 + T^2 \sum_{j,n=1}^{N} p^j L^j p^n L^n \sum_{k,l=1}^{K} a_k^j a_l^n \mathbf{Cov}[R^k, R^l] \tag{100}$$

Proof.
In general holds:

$$\Phi_X(s) = \mathbf{E}[\Phi_{X|R}(s)] \tag{101}$$

To determine the nth moment one needs to compute:

$$\mathbf{E}[X^n] = \lim_{s\to 0} \frac{1}{i^n}\frac{d^n}{ds^n}\mathbf{E}[\Phi_{X|R}(s)] \tag{102}$$

In fact, there are three limits in this expression: the expectation (integration), a differentiation and the valuation at one point. From (97) it is clear, that $\Phi_{X|R}(s)$ is analytical and that it may written as a power series in s. Since power series converge uniformly on a compact interval, the exchange of the limits is allowed:

$$\mathbf{E}[X^n] = \mathbf{E}[\lim_{s\to 0} \frac{1}{i^n}\frac{d^n}{ds^n}\Phi_{X|R}(s)] \tag{103}$$

The first two derivatives of $\Phi_{X|R}(s)$ are

$$\Phi'_{X|R}(s) = \Phi_{X|R}(s) \cdot \sum_{j=1}^{N} p^j T \left(a_0^j + \sum_{k=1}^{K} a_k^j R^k\right) e^{iL^j s}(iL^j) \tag{104}$$

$$
\begin{aligned}
\Phi''_{X|R}(s) &= \Phi_{X|R}(s) \cdot \sum_{j=1}^{N} p^j T \left(a_0^j + \sum_{k=1}^{K} a_k^j R^k \right) e^{iL^j s} (iL^j)^2 && (105)\\
&\quad + \Phi_{X|R}(s) \cdot \left[\sum_{j=1}^{N} p^j T \left(a_0^j + \sum_{k=1}^{K} a_k^j R^k \right) e^{iL^j s} (iL^j) \right]^2 && (106)
\end{aligned}
$$

Valuation at $s = 0$ yields:

$$
\begin{aligned}
\frac{1}{i} \Phi'_{X|R}(0) &= T \sum_{j=1}^{N} p^j L^j (a_0^j + \sum_{k=1}^{K} a_k^j R^k) && (107)\\
\frac{1}{i^2} \Phi''_{X|R}(0) &= T \sum_{j=1}^{N} p^j (L^j)^2 (a_0^j + \sum_{k=1}^{K} a_k^j R^k) + \left[T \sum_{j=1}^{N} p^j L^j (a_0^j + \sum_{k=1}^{K} a_k^j R^k) \right]^2 && (108)
\end{aligned}
$$

Take the expectation over R and recall that $\mathbf{E}[R^k] = 1$. To simplify the expressions, the relation $a_0^j + \sum_{k=1}^{K} a_k^j = 1$ has been used:

$$
\begin{aligned}
\mathbf{E}[X] &= T \sum_{j=1}^{N} p^j L^j && (109)\\
\mathbf{E}[X^2] &= T \sum_{j=1}^{N} p^j (L^j)^2 + T^2 \left[\sum_{j=1}^{N} p^j L^j \right]^2 \\
&\quad + T^2 \sum_{j,n=1}^{N} p^j L^j p^n L^n \sum_{k,l=1}^{K} a_k^j a_l^n \mathbf{Cov}[R^k, R^l] && (110)
\end{aligned}
$$

■

Remark 8 *Note that in the case of uncorrelated sector variables the first two moments are identical to the moments in the CreditRisk$^+$ model, since the variance of the random variables R^k is given by $\sigma_k^2 = \mathbf{Cov}[R^k, R^k]$.*

4.3 The Characteristic Function by a Monte Carlo Approach

From equation (97) the characteristic function of X conditioned on R is known and hence the characteristic function of X can be written as an expectation:

$$
\begin{aligned}
\Phi_X(s) &= \mathbf{E}[\Phi_{X|R}(s)] && (111)\\
&= e^{\left(\sum_{j=1}^{N} p^j T a_0^j (e^{iL^j s} - 1) \right)} \mathbf{E} \left[\exp \left(\sum_{k=1}^{K} R^k \sum_{j=1}^{N} T p^j a_k^j (e^{iL^j s} - 1) \right) \right] && (112)\\
&= \exp\left(\xi_0(s)\right) \ \mathbf{E} \left[\exp \left(\sum_{k=1}^{K} R^k \xi_k(s) \right) \right] \qquad \text{where} && (113)\\
\xi_k(s) &:= \sum_{j=1}^{N} T p^j a_k^j (e^{iL^j s} - 1) \qquad \text{for } k = 0, \dots K && (114)
\end{aligned}
$$

First one has to decide, for which s one wants to compute the characteristic function $\Phi_X(s)$ and S denotes the number of sample points. Since the functions $\xi_k(s)$ are deterministic functions, this functions can be valuated before starting the Monte Carlo loop for each regarded s. The advantage of this organisation of the Monte Carlo procedure is, that each diced tuple of R can be used for each s to compute the expectation, hence the computation time for generation of the random numbers is independent of S. In the case that R^k are dependent lognormal distributed, the effort to dice one sample is given by $\mathcal{O}(K^2)$, however, the same effort will mostly hold for other multivariate distributions. If M is the number of Monte Carlo iterations, the computational effort of this method is given by:

$$\underbrace{\mathcal{O}(KNS)}_{\text{Computing } \xi_k} + \underbrace{\mathcal{O}(MK^2)}_{\text{Dicing } R^k} + \underbrace{\mathcal{O}(MSK)}_{\text{Valuation of the Monte Carlo sum}} \tag{115}$$

Note, that in practice N and M are larger than K, S and that the computational effort does not contain a term NM. So the effort of this Monte Carlo method is tolerable, in contrast to a direct simulation. A direct Monte Carlo approach on the model for X would effort for each Monte Carlo simulation first to sample the sector variables R, then compute the default intensities and sample the time of default of each obligor. Hence the direct simulation approach is not feasible due to a computational effort of $\mathcal{O}(KMN)$.

The proposed algorithm admits two possibilities for parallel computing. The first possibility is the usually Monte Carlo parallelism of valuating the expression parallel for several diced random numbers. Second, one can also compute independently the expression for each s and fixed dice of random numbers R. Since one is interested in a expectation and not in the simulation of a stochastic process, it may be favourable to use Quasi-Monte-Carlo, due to a potentially much better convergence.

Since the first two moments of X are known and a numerical algorithm to obtain the characteristic function of X is established, one can compute the distribution of X by one of the Fourier inversion techniques presented in section 3. Using the method based on FFT, one only has to choose the number of sample points S and then this method determines the sample points s, since the sample interval in the X space is naturally given by $[0, \sum_{j=1}^{N} L_j]$. Since the first two moments of X are known, one may also perform a Fourier inversion based on theorem 4. Then one has the freedom to select an arbitrary Fourier integration method and the sample points s of the characteristic function will be determined by the choice of the integration method.

5 A time continuous model

In the previous models the distribution of the loss for a fixed date T was studied. In this part, a model is introduced, which allows the credit risk analysis in continuous time. This model is abutted to the model with dependent sector variables; but to describe the sectors continuous time processes are involved rather than discrete random variables.

5.1 Introduction of the model

Let us fix a time horizon $T_\infty > 1$ and introduce K sector processes R_t^k on $t \in [0, T_\infty]$ which are assumed to be geometric Brownian motions with:

$$\begin{aligned} R_0^k &= 1 && (116)\\ \mathbf{E}[R_t^k] &= 1 && (117)\\ \mathbf{Cov}[\ln R_t^k, \ln R_t^l] &= C_{kl} t && (118) \end{aligned}$$

As before, consider sector affiliations a_k^j, such that the following relations hold:

$$\begin{aligned} a_k^j &\geq 0 && \forall\, j = 1, \ldots, N;\; k = 0, \ldots, K && (119)\\ \sum_{k=0}^{K} a_k^j &= 1 && \forall\, j = 1, \ldots, N && (120) \end{aligned}$$

Again there is a rating parameter p^j for each obligor, which is the probability that the jth obligor defaults within one year and again it is assumed that p^j is rather small. For the interpretation of this parameter, the proof of lemma 1 holds. In contrast to the previous models, the default intensity becomes a stochastic process now:

$$\lambda_t^j := p^j \left(a_0^j + \sum_{k=1}^{K} a_k^j R_t^k \right) =: p^j \tilde{\lambda}_t^j \qquad (121)$$

The default event of the jth obligor is denoted by T^j and T^j is exponentially distributed with the intensity process λ_t^j. Also the processes I_t^j are introduced:

$$I_t^j := \begin{cases} 0 & \text{if } t < T^j \\ 1 & \text{if } t \geq T^j \end{cases} \qquad (122)$$

Conditioned on R, these processes are independent from the sector processes and

$$P[T^j < t] = P[I_t = 1] = 1 - \exp\left(- \int_0^t \lambda_s ds \right) \qquad (123)$$

If the jth obligor defaults, the loss due to this default is given by L^j and hence the process of cumulated defaults is given by

$$X_t := \sum_{j=1}^{N} L^j I_t^j \qquad (124)$$

5.1.1 The Default Indicator Process I_t

The default indicator process defined in this section is the natural extension from the indicator variable defined in sections 2 and 4. To understand this, the relation between the default process I_t and the indicator variable from sections 2 and 4 is analyzed. In those models, the default time was given by an exponential distributed random number and $P[I_T = 1] = 1 - e^{-\lambda T}$ Apply this idea to infinitesimal small time intervals of size h. Then the probability that a default occurs up $t + h$ is given by the probability that there is a default before t plus the probability of default between t and $t+h$:

$$P[I_{t+h}=1] \quad = \quad P[I_t=1]+(1-P[I_t=1])(1-e^{-\lambda_t h}) \tag{125}$$

Hence:

$$\frac{P[I_{t+h}=1]-P[I_t=1]}{h} \quad = \quad (1-P[I_t=1])(\lambda_t+\mathcal{O}(h)) \tag{126}$$

In the limit $h \to 0$:

$$\frac{d}{dt}P[I_t=1] \quad = \quad (1-P[I_t=1])\lambda_t \tag{127}$$

This is a ordinary differential equation which has to be solved under the condition $P[I_0=1]=0$. Since its solution is given by (123), (123) is the natural extension of the default indicator to continuous time.

Since I_t is a jump process, one may also be interested in the relationship between I_t and the Poisson process:

Lemma 8 (Default indicator and Poisson process) *Let Y_t be a Poisson process with time dependent default intensity $\lambda_t \geq 0$, that is a process with right continuous paths and existing left limits such that for all $t > s \geq 0$ holds:*

$$\begin{aligned}
Y_0 &= 0 && (128)\\
Y_t - Y_s &\in \mathbb{N} && (129)\\
P[Y_t - Y_s = n] &= \frac{1}{n!}\left(\int_s^t \lambda_u du\right)^n \cdot \exp\left(-\int_s^t \lambda_u du\right) && (130)\\
Y_t - Y_s & \quad \textit{is independent of } Y_{[0,s]} && (131)
\end{aligned}$$

Then the following relation holds for the default indicator process

$$I_t \quad = \quad \begin{cases} 0 & \textit{if } Y_t = 0 \\ 1 & \textit{if } Y_t > 0 \end{cases} \tag{132}$$

5.2 The characteristic function of X_t by Monte Carlo

Let $\mathcal{R}_t$ be the natural filtration induced by the processes R^k. If one defines the sector variables in section 4 to be $\frac{1}{t}\int_0^t R^k_\tau \, d\tau$, one obtains the characteristic function of X_t conditioned on $\mathcal{R}_t$:

$$\Phi_{X_t|\mathcal{R}}(s) \quad = \quad \exp\left(\sum_{j=1}^N (e^{iL^j s}-1)p^j(a_0^j t + \sum_{k=1}^K a_k^j \int_0^t R^k_\tau d\tau)\right) \tag{133}$$

Since the sector processes R^k_t are not independent, the expectation to get the characteristic function $\Phi_{X_t}(s)$ can be valuated by Monte Carlo only:

$$\begin{aligned}
\Phi_{X_t}(s) &= \mathbf{E}[\Phi_{X_t|\mathcal{R}}(s)] && (134)\\
&= \exp\left(\sum_{j=1}^{N}(e^{iL^j s}-1)p^j a_0^j t\right)\mathbf{E}\left[\exp\left(\sum_{k=1}^{K}\sum_{j=1}^{N}(e^{iL^j s}-1)p^j a_k^j \int_0^t R_\tau^k d\tau\right)\right]\\
&= \exp(\xi_0(s)t)\ \mathbf{E}\left[\exp\left(\sum_{k=1}^{K}\xi_k(s)\int_0^t R_\tau^k d\tau\right)\right] \qquad \text{where} && (135)\\
\xi_k(s) &:= \sum_{j=1}^{N}(e^{iL^j s}-1)p^j a_k^j \qquad \text{for } k=0,\dots K && (136)
\end{aligned}$$

Let $\mathcal{S}$ be the set of values of s for which one wants to valuate the characteristic function. Then it is possible to compute the expressions $\xi_k(s)$ $s \in \mathcal{S}$ before starting the Monte Carlo procedure. For each Monte Carlo iteration one has to sample the random numbers

$$I^k \quad \sim \quad \int_0^t R_\tau^k d\tau \tag{137}$$

with dependent geometric Brownian motions R_τ^k. Using one draw of I^k one computes $\sum_{k=1}^{K}\exp(\xi_k(s)I^k)$ for each $s \in \mathcal{S}$ and the total cost of this Monte Carlo method is comparable with the method in section 4.3. There is only a larger effort in dicing I^k, since one has to sample paths and to integrate them instead of dicing one lognormal random number, but the asymptotic costs are again given by (115).

Remark 9 *To sample I^k one can proceed as follows. Let B be the Cholesky decomposition of the log–covariance matrix C and W_t^l independent Brownian motions. Then R_t^k is given by*

$$R_t^k \quad = \quad \exp\left(\sum_{l=1}^{K} B_{kl}W_t^l - \frac{1}{2}C_{kk}t\right) \tag{138}$$

To compute the integral one divides the interval $[0,t]$ into D intervals of equal length and computes the integral by the approximation

$$I^k \quad \approx \quad \frac{t}{2D}\sum_{i=1}^{D}(R^k_{\frac{(i-1)t}{D}} + R^k_{\frac{it}{D}}) \tag{139}$$

There is a lot of literature on the properties of the integral of the geometric Brownian motion, see for example [6, 7, 9, 14].

In order to obtain the distribution of X one can use the Fourier inversion techniques presented in section 3. Usually one would use the FFT based method in this context, since then the set $\mathcal{S}$ is clearly given by the discretization of the characteristic function. To use the Fourier inversion method based on theorem 4 one needs to know the first two moments of X, which will be computed now to round off the discussion.

5.3 Moments of X_t

Lemma 9 *Let R_t^k and R_t^l be two geometric Brownian motions where $(\ln R_t^k + \frac{1}{2}C_{kk}t)$ and $(\ln R_t^l + \frac{1}{2}C_{ll}t)$ are normal distributed with mean 0 and covariance $C_{kl}t$. Then holds:*

$$\mathbf{E}\left[\int_0^t R_\tau^k d\tau\right] = t \tag{140}$$

$$\mathbf{E}\left[\int_0^t R_\tau^k d\tau \int_0^t R_\theta^l d\theta\right] = \frac{2}{(C_{kl})^2}\left(e^{C_{kl}t} - 1 - C_{kl}t\right) \tag{141}$$

Proof. To prove the first equation, one only has to use Fubini's theorem. For the second statement, let $\tau \leq \theta$:

$$\mathbf{E}[R_\tau^k R_\theta^l] = \mathbf{E}[R_\tau^k R_\tau^l \frac{R_\theta^l}{R_\tau^l}] = \mathbf{E}[R_\tau^k R_\tau^l] = \exp(C_{kl}\tau) \tag{142}$$

Hence,

$$\mathbf{E}\left[\int_0^t R_\tau^k d\tau \int_0^t R_\theta^l d\theta\right] = \int_0^t d\tau \int_0^t d\theta \mathbf{E}\left[R_\tau^k R_\theta^l\right] \tag{143}$$

$$= 2\int_0^t d\tau \int_\tau^t d\theta e^{C_{kl}\tau} = 2\int_0^t d\tau e^{C_{kl}\tau}(t-\tau) \tag{144}$$

$$= \frac{2}{(C_{kl})^2}\left(e^{C_{kl}t} - 1 - C_{kl}t\right) \tag{145}$$

■

This lemma and theorem 5 prove the

Theorem 6 *The expectation and variance of X_t are given by*

$$\mathbf{E}[X_t] = t\sum_{j=1}^N p^j L^j \tag{146}$$

$$\mathbf{Var}[X] = t\sum_{j=1}^N p^j (L^j)^2 + \sum_{j,n=1}^N p^j L^j p^n L^n \sum_{k,l=1}^K a_k^j a_l^n \left(\frac{2(e^{C_{kl}t} - 1 - C_{kl}t)}{(C_{kl})^2} - t^2\right) \tag{147}$$

6 Combining Market Risk and Credit Risk

Up to now only credit risk has been analyzed in this paper. In this section the previous model will be extended in such a way, that it incorporates market risk. The idea of the approach presented here differs from other models, which combine market and credit risk (see e.g. [5]). It starts from the CreditRisk$^+$ model which is

widen in such a way, that when there is no credit risk one obtains the well known Delta normal approach to assess market risk.
Whenever one talks about market risk, there is a portfolio depending on market risk factors and one has the possibility to evaluate a portfolio for a given state of these factors. Here a model of a portfolio which combines market risk and credit risk is introduced. Finally, the computation of the profit and loss distribution (P&L) of this portfolio is aspired and an approximative solution of this task can be given.
In addition to the assumption that default probabilities are small, it is also assumed – according to the Delta normal model – that the effect of the (normal distributed) market fluctuations may be linearized. An algorithm to compute the P&L is presented which is based on these assumptions. The computational effort of this algorithm is tolerable for a medium number of obligors and is independent of the number of market risk factors.

6.1 A Portfolio with Market Risk and Credit Risk

The credit risk driving factors are the default indicators I_t^j and the sector processes R_t^k of the credit risk model presented in section 5. Due to changes of the sector processes R_t^k, the obligors default intensities alter and this effects the probability of default of each obligor. In fact, this is the so–called spread risk, which is modelled by the sector processes in this way.
Additional to the sector variables, there are market risk factors M_t^l, $l = 1, \dots, \tilde{K}$. Examples of market risk factors are returns of stock prices, foreign exchange rates or interest rates. Let M_t^l are dependent Brownian motions with drift 0 and different volatilities. To describe the dependencies between R_t^k and M_t^l a covariance matrix C of the following structure is introduced:

$$C = \begin{pmatrix} C^1 & C^{2\top} \\ C^2 & C^3 \end{pmatrix} \tag{148}$$

C^1 is a $K \times K$ matrix, C^3 is a $\tilde{K} \times \tilde{K}$ matrix and C^2 is a $\tilde{K} \times K$ matrix and the entries of these matrices are given by

$$C_{kl}^1 = \frac{1}{t}\mathbf{Cov}(\ln R_t^k, \ln R_t^l) \tag{149}$$

$$C_{kl}^2 = \frac{1}{t}\mathbf{Cov}(M_t^k, \ln R_t^l) \tag{150}$$

$$C_{kl}^3 = \frac{1}{t}\mathbf{Cov}(M_t^k, M_t^l) \tag{151}$$

Let $N^j(M_t, t)$ denote the nominal amount of the jth obligor at time t, discounted by the risk free interest rate. For simplicity, let us assume that there is only one and fixed settlement date T^j for each obligor. The probability, that the jth obligor survives up to T^j under the condition that no default has occured up to time t is given by $P[I_{T^j}^j = 0 | I_t^j = 0]$. These are functions of the state of the sector processes:

$$q^j(r,t) := P[I_{T^j}^j = 0 | R_t = r \wedge I_t^j = 0] \tag{152}$$

$$= \mathbf{E}\left[\exp\left(-p^j \int_t^{T^j} (a_0^j + \sum_{k=1}^{K} a_k^j R_\tau^k) d\tau\right) \middle| R_t^k = r^k\right] \tag{153}$$

Since any claim fraught with credit risk is discounted by this probability, the value process of the portfolio can be written:

$$V_t \;=\; \sum_{j=1}^{N} N^j(M_t,t)q^j(R_t,t)(1-I_t^j) \tag{154}$$

6.2 Obtaining an approximative P&L

Besides the assumption that p^j is small, also the following linearization is used:

Assumption 2 (Delta Approach) *Let us assume, that the nominal functions may be approximated by linear functions:*

$$N^j(M_t,t) \;\approx\; N^j(M_0,0)+\frac{\partial N^j}{\partial t}(M_0,0)t+\frac{\partial N^j}{\partial M_t}(M_0,0)\cdot M_t \tag{155}$$

$$=: \; \overset{0}{N^j}(t)+\overset{\Delta}{N^j}\cdot M_t \tag{156}$$

Using this assumption, the value process is given by:

$$V_t \;=\; \sum_{j=1}^{N}(\overset{0}{N^j}(t)+\overset{\Delta}{N^j}\cdot M_t)\; q^j(R_t,t)\;(1-I_t^j) \tag{157}$$

In the next step the dependence between M_t and R_t will be analyzed. Let B be the Cholesky decomposition of the covariance matrix C, with the structure:

$$C=\begin{pmatrix} C^1 & C^{2\top} \\ C^2 & C^3 \end{pmatrix} \;=\; \begin{pmatrix} B^1 & 0 \\ B^2 & B^3 \end{pmatrix}\begin{pmatrix} B^1 & 0 \\ B^2 & B^3 \end{pmatrix}^{\top}=BB^{\top} \tag{158}$$

Then the usual method to sample $\ln R_t$ and M_t is to choose two vectors W_t^1, W_t^2 of independent Brownian motions and to use the transformation

$$\begin{pmatrix} \ln R_T^k+\frac{1}{2}C_{kk}t \\ M_t \end{pmatrix} \;=\; \begin{pmatrix} B^1 & 0 \\ B^2 & B^3 \end{pmatrix}\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \tag{159}$$

Hence the state of the sector process is given by

$$R_t^k \;=\; \exp\left((B^1W_t^1)^k-\frac{1}{2}C_{kk}^1 t\right) \tag{160}$$

and M_t can be expressed by

$$M_t \;=\; B^2W_t^1+B^3W_t^2 \tag{161}$$

Hence, the portfolio has the representation

$$V_t \;=\; \sum_{j=1}^{N} q^j(R_t,t)(1-I_t^j)\left(\overset{0}{N^j}(t)+B^{2\top}\overset{\Delta}{N^j}\cdot W_t^1+B^{3\top}\overset{\Delta}{N^j}\cdot W_t^2\right) \tag{162}$$

The P&L may be determined by Fourier inversion of the characteristic function of V_t. A first step to compute the characteristic function is to determine the characteristic function conditioned on the state of R_t and I_t:

$$\Phi_{V_t|(R_t,I_t)}(s) = \mathbf{E}[e^{isV_t}|R_t,I_t] \tag{163}$$

$$= \exp\left(is\sum_{j=1}^{N} q^j(R_t,t)(1-I_t^j)(\overset{0}{N^j}(t)+\overset{\Delta}{N^j}B^2W_t^1)\right)\times$$
$$\exp\left(-\frac{1}{2}s^2\left\|\sum_{j=1}^{N} q^j(R_t,t)(1-I_t^j)B^{3\top}\overset{\Delta}{N^j}\right\|^2 t\right) \tag{164}$$

As in the whole paper, the default probability for each obligor is assumed to be rather small. Hence by lemma 2 one may approximate:

$$q^j(r,t) \approx \exp\left(-p^j(T^j-t)(a_0^j+\sum_{k=1}^{K}a_k^j r^k)\right) \tag{165}$$

The characteristic function of V_t can now be evaluated by a Monte Carlo method using the representation

$$\Phi_{V_t}(s) = \mathbf{E}\left[\Phi_{V_t|(\mathcal{R}_t,\mathcal{I}_t)}(s)\right] \tag{166}$$

For one Monte Carlo sample one has to sample the paths W_t^1 of independent Brownian motions. By equation (160) one can determine R_t^k and then by equation (165) one obtains $q^j(R_t,t)$. Then for each obligor, one has to sample the binary random variable

$$I_t^j = \begin{cases} 0 & \text{with probability } 1-\exp\left(-p^j\int_0^t(a_0^j+\sum_{k=1}R_\tau^k)d\tau\right) \\ 1 & \text{with probability } \exp\left(-p^j\int_0^t(a_0^j+\sum_{k=1}R_\tau^k)d\tau\right) \end{cases} \tag{167}$$

Of course, with one sample of I_t and R_t one can evaluate the expression under the expectation for several s. If one proceeds so, the numerical effort of this method is obviously of order $\mathcal{O}(M\cdot(K+N+N\cdot K+S))$ where M denotes the number of Monte Carlo valuations and S is the number of sample points of $\Phi_{V_t}(s)$. Hence this method can be used in practice if the number of obligors is not too big; but since the effort is independent of the number of market risk factors $\tilde{K}$, this approach can be applied even for large $\tilde{K}$. In order to obtain the density of the portfolios profit and loss distribution one can use the Fourier inversion technique based on the Fast Fourier Transformation described in section 3.2.

6.3 Concluding Remarks

A model which combines market risk and credit risk has been presented. In the special case, that there is only one obligor who never defaults, the previous analysis of the P&L distribution gives the so–called "Delta normal" approach, which is the most simple and well known idea to deal with market risk alone. In the case, that

there is no market risk, the model conforms with the credit risk model presented in section 5. This underlines that the presented model is a natural generalization to describe market risk and credit risk.
In order to find other and more general models which combine market risk and credit risk it is necessary to find other credit risk models which allow the handling of a stochastic loss given default. One approach in this direction has been done by Bürgisser et. al. [3]. They assume a stochastic independence between the sector variables and the processes, which effect the amount of the loss given default. They also give a remark how to overcome this assumption, but a deeper analysis remains to be done. In [5] another approach is presented to model market and credit risk. Duffie and Pan split the portfolio into a *value component* and a *default component* and they treat each component separately.
For applications in risk management it seems to be necessary to model dependencies between the sectors and the loss given default. For example, assume that you hold put options on the stock of a bank which are sold by this bank. In the case of default of this bank the stock will fall, but your (theoretical) win on the puts is worthless since the bank is unable to pay them out.
Based on the density of the portfolios P&L one can perform much more computations. The most risk measures used in practice are functionals of the P&L which can be determined, since the Fourier inversion yields the whole distribution of a portfolio. The same argument also holds for the pricing of derivatives on such portfolios. For example, the price of a European option can be represented as an expectation of a function of the portfolio return. There are plain credit risk portfolios traded, e.g. Asset backed securities, but it seems very likely that options on more complex portfolios will be sold in future and hence there is a demand on models which unify market and credit risk.

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