Integrating Interest Rate Risk

in Credit Portfolio Models

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Abstract:

A typical shortcoming of most current credit portfolio models is the lack of a stochastic modeling of risk factors, such as interest rates or credit spreads, during the revaluation process at the risk horizon. Within the simple credit risk model underlying the Internal Ratings-based approach of Basel II with incorporated correlated interest rate risk the effect which results from neglecting the stochastic nature of market risk factors is shown for an infinitely large, homogeneous portfolio of defaultable coupon bonds. The consequence of ignoring interest rate risk can be a significant underestimation of the economic capital needed as a protection against unexpected losses. The lower the correlation of the firms' asset returns, the lower the unconditional default probability and the longer the bonds' time to maturity, the higher is the difference between the VaR with and without considering interest rate risk during the revaluation process at the risk horizon.

Key words: credit risk, interest rate risk, credit portfolio model, IRB-approach

JEL classification: G 21, G 33

Integrating Interest Rate Risk in Credit Portfolio Models

I. Indroduction

A typical shortcoming of most current credit portfolio models is the lack of a stochastic modeling of risk factors, such as interest rates or credit spreads, during the revaluation process at the risk horizon. For example, fixed income instruments, such as bonds or loans, are revalued at the risk horizon using the current forward rates and (rating class specific) forward credit spreads for discounting future cash flows. Hence, the stochastic nature of the instrument's value in the future which results from changes in factors other than credit quality is ignored and the riskiness of the credit portfolio at the risk horizon is underestimated. A further consequence is that correlations between changes of the debtor's default probability and changes of market risk factors and, hence, the exposure at default cannot be integrated into the credit portfolio model. This drawback is especially relevant for portfolios of defaultable market-driven derivatives. One reason why risk factors not being directly related to credit risk are neglected in most current credit portfolio models is that there is still no common sense how to model the credit quality of a debtor and the dependencies between the credit quality of different debtors. Hence, it might be over-ambitious to incorporate correlations between market risk factors and the credit quality, especially as the empirical findings even concerning the sign of the correlation are not unambiguous. Additionally, this would increase the computational burden for calculating risk measures of realistic credit portfolios significantly.

Related papers which analyze the effect of integrating a further risk factor, such as stochastic interest rates or stochastic credit spreads, into a credit portfolio model are from *Kiesel*, *Perraudin* and *Taylor* (1999), *Kijima* and *Muromachi* (2000), *Barth* (2000), *Barnhill* and *Maxwell* (2002) and *Grundke* (2002).[1] *Grundke* (2002) extends the credit risk model underlying the Internal Ratings-Based (IRB) approach proposed by the Basle Committee on Banking Supervision[2] by uncorrelated interest rate risk and applies this model framework on a homogenous portfolio of defaultable zero coupon bonds. In this paper, the framework of *Grundke* (2002) is modified by making use of the assumption that the portfolio is infinitely large, by modeling credit and interest rate risk as correlated risks and by applying this model on a portfolio of defaultable coupon bonds. As in *Grundke* (2002) no Monte Carlo-simulations are needed. In section III the results are compared with those of *Kiesel*, *Perraudin* and *Taylor* (1999) and *Grundke* (2002) because their model set-ups are most closely related to the approach of this paper.

This paper is structured as follows: Section II consists of a description of the relevant assumptions of the credit portfolio model and derives the probability distribution of the credit portfolio value at the risk horizon. In section III this model is applied to a homogenous portfolio of defaultable coupon bonds, and the influence of changing one of the parameters on the credit portfolio's risk measures and on the effect resulting from integrating interest rate risk is analyzed. The paper concludes in section IV with a summary of the results and a short discussion of possible extension of the analysis.

II. The Model

It is assumed that the credit portfolio consists of N coupon bonds with identical face value F, maturity date T, coupon c and coupon dates $H \leq t_1 < \ldots < t_M = T$ issued by N different corporates. The risk horizon H of the credit portfolio model is one year. P denotes the real world probability measure.

It is further assumed that the return of the firms' assets can be explained by the following two-factor model:[3]

(1) $R_j = \omega_1 Z + \omega_2 X_r + \omega_3 \varepsilon_j$ ($j \in \{1,\dots,N\}$,

with

$Z, X_r, \varepsilon_j \sim i.i.d.\, N(0,1)$ ($j \in \{1,\dots,N\}$),

$Cov(Z, X_r) = 0$,

$Cov(Z, \varepsilon_j) = 0$ ($j \in \{1,\dots,N\}$),

$Cov(X_r, \varepsilon_j) = 0$ ($j \in \{1,\dots,N\}$).

With Z a macroeconomic factor representing systematic credit risk is denoted, the ε_j 's stand for firm specific or idiosyncratic risk, and X_r is a stochastic factor driving the term structure of risk-free interest rates, which is assumed to evolve according to the term structure model of *Vasicek* (1977):

(2) $dr(t) = \kappa(\theta - r(t))\, dt + \sigma dW(t)$

($\kappa, \theta, \sigma \in \mathbb{R}_{++}$, $W(t)$ standard-Brownian motion under *P*).

(2) is a mean-reverting process because $(r(t))_{t \in \mathbb{R}_+}$ always tends back to the level θ; the higher the value κ the more unlikely are deviations from this level. The standard-Brownian motion $(W(t))_{t \in \mathbb{R}_+}$ and the random variables Z and ε_j ($j \in \{1,\dots,N\}$) are assumed to be mutually independent. The probability distribution of $r(t)$ given the information F_s of the process up to time $s \le t$ is:[4]

(3) $r(t)\big|_{F_s} \sim N\left(\mu_r(s,t), \sigma_r^2(s,t)\right)$

with

(4) $\mu_r(s,t) := \theta + (r(s) - \theta) e^{-\kappa(t-s)}$ and

(5) $\sigma_r^2(s,t) := \dfrac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(t-s)})$.

Hence, the risk-free interest rate $r(H)$ at the risk horizon can be represented as:

(6) $$r(H)\big|_{F_0} = \underbrace{\theta + (r(0)-\theta)e^{-\kappa H}}_{=E^P[r(H)]} + \sqrt{\frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa H}\right)}X_r$$

with $X_r \sim N(0,1)$, which enters into the definition (1) of the two-factor model for the firms' asset returns. With the specification (1) of the asset return model credit risk and interest rate risk are correlated risk factors.

As $r(t)$ is normally distributed negative interest rates are – usually only with a small probability – possible. Unfortunately, this is not the only drawback of the Vasicek-model: It is not possible to adapt the model perfectly to a given current term structure of interest rates. Nevertheless, this term structure model is chosen here for the sake of simplicity. But the use of alternative term structure models[5] should not change the qualitative results of this paper.

The value $p(t,T)$ of a risk-free zero coupon bond with face value F and maturity date T at time $t \in [0,T]$ is given in the Vasicek-model by:[6]

(7) $$p\big(r(t),t,T\big) = Fe^{\left(\frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right)\left(R(\infty)-r(t)\right)-(T-t)R(\infty)-\frac{\sigma^2}{4\kappa^3}\left(1-e^{-\kappa(T-t)}\right)^2\right)},$$

or, equivalently, using (6), by

(8) $$p\big(X_r,t,T\big) = Fe^{\left(\frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right)\left(R(\infty)-\left(\theta+(r(0)-\theta)e^{-\kappa t}+\sqrt{\frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa t}\right)}X_r\right)\right)-(T-t)R(\infty)-\frac{\sigma^2}{4\kappa^3}\left(1-e^{-\kappa(T-t)}\right)^2\right)}$$

with

(9) $$R(\infty) = \theta + \lambda\frac{\sigma}{\kappa} - \frac{1}{2}\frac{\sigma^2}{\kappa^2}.$$

$R(\infty)$ denotes the return of a risk-free zero coupon bond with infinite time to maturity, and $\lambda \in \mathbb{R}_+$ is the constant market price of interest rate risk.

The sensitivities $\omega_1 \in \mathbb{R}_{++}$, $\omega_3 \in \mathbb{R}_+$, $\omega_2 \in \mathbb{R}$ to the risk factors are assumed to be identical to all debtors which implies an identical correlation ρ between all pairs of asset returns and between all pairs of asset returns and the risk-free interest rate. Without loss of generalization the variance of the asset returns R_j can be normalized to one:

$$Var(R_j) = \omega_1^2 + \omega_2^2 + \omega_3^2 \overset{!}{=} 1$$

(10) $$\Leftrightarrow \omega_3 = \sqrt{1-(\omega_1^2+\omega_2^2)}\ .$$

For the sensitivities ω_1 and ω_2 we get:

$$\rho = Corr(R_i, R_j) = Cov(R_i, R_j) = Cov(\omega_1 Z, \omega_1 Z) + Cov(w_2 X_r, w_2 X_r) = \omega_1^2 + \omega_2^2$$

(11) $$\Leftrightarrow \omega_1 = \sqrt{\rho - \omega_2^2}\ ,$$

and

(12) $$\omega_3 = \sqrt{1-\rho}\ .$$

There is a default of the debtor j if the asset return R_j is below a critical level α at the risk horizon. The value of the parameter α is assumed to be equal for all debtors and can be calculated by the relationship

$$q = P(R_j \le \alpha) = \Phi(\alpha)$$

(13) $$\Leftrightarrow \Phi^{-1}(q) = \alpha$$

where $\Phi^{-1}(\cdot)$ is the inverse of the cumulative density function of the standard normal distribution, and q denotes the unconditional one year default probability of a certain rating class taken for example from the transition matrices published by Moody's or Standard & Poor's.

The one-year probability of default of debtor j conditional on the realization of the systematic risk factor Z and the interest rate factor X_r is given by:

$$P\left(R_j \leq \alpha \middle| Z = z, X_r = x_r\right) = P\left(\varepsilon_j \leq \frac{\alpha - \omega_1 Z - \omega_2 X_r}{\sqrt{1-\rho}} \middle| Z = z, X_r = x_r\right)$$

(14) $$= \Phi\left(\frac{\alpha - \omega_1 z - \omega_2 x_r}{\sqrt{1-\rho}}\right) =: q(z, x_r) .$$

As (14) shows, the specification (1) of the multi-factor model for the individual asset returns implies that the conditional probability of default and the term structure of risk-free interest rates are correlated. The degree of correlation is determined by the value of the sensitivity ω_2: The larger the absolute amount of this value is, the higher is the influence of the risk-free interest rate $r(H)$ on the asset return R_j and, hence, on the conditional default probability.

Conditional on the realizations of the random variables Z and X_r the asset returns and, hence, the default events of different debtors are independent. That is why the (strong) law of large numbers[7] can be applied, which ensures that with the number of debtors approaching infinity the fraction D of debtors that actually defaults equals almost surely the individual conditional default probability $q(z, x_r)$:

$$P\left(\lim_{N\to\infty}\left(\underbrace{\frac{\sum_{j=1}^{N} 1_{\{R_j \leq \alpha\}}}{N}}_{=:D} - E^P\left[\frac{\sum_{j=1}^{N} 1_{\{R_j \leq \alpha\}}}{N}\right]\right) = 0 \middle| Z = z, X_r = x_r\right) = 1$$

(15) $\Rightarrow P\left(D = q(z, x_r) \middle| Z = z, X_r = x_r\right) = 1$.

The price $v_j(X_r, H, T, c)$ of a coupon bond with face value F, maturity date T, coupon c and coupon dates $H \le t_1 < \ldots < t_M = T$ at the risk horizon whose issuer j has not yet defaulted (*nd*) up to this time is given by:

(16) $$v_j^{nd}(X_r, H, T, c) = \sum_{m=1}^{M} c e^{-(R(X_r, H, t_m) + S(0, H, t_m))(t_m - H)} + F e^{-(R(X_r, H, T) + S(0, H, T))(T - H)}$$

where $R(X_r, H, t_m) = -(t_m - H)^{-1} \ln(p(X_r, H, t_m)/F)$ denotes the stochastic risk-free discount factor for the time interval $[H, t_m]$, and $S(0, H, t_m)$ is the forward credit spread for the time interval $[H, t_m]$ observed at $t = 0$. Of course, the simplifying assumption of non-stochastic credit spreads is questionable because it is not too plausible that real-world default probabilities are stochastically varying over time but credit spreads remain deterministic. But as introducing stochastic credit spreads depending for example on the realized asset return at the risk horizon would complicate the analysis this assumption is maintained. It should be remarked that also in the commercial credit portfolio model CreditPortfolioView™ stochastic transition probabilities and non-stochastic credit spreads (for each rating class) are used simultaneously.[8]

If an issuer j has defaulted (*d*) within the risk horizon it is assumed that the bondholder gets a fraction $\delta \in [0,1]$ of the face value F of the coupon bond at the risk horizon H irrespective of the coupon or the remaining time to maturity:

(17) $v_j^d = \delta F$.

The value of the parameter $\delta \in [0,1]$ can vary with the seniority of the coupon bond.

The value $\Pi(H)$ of the portfolio of corporate coupon bonds at the risk horizon is:

$$\Pi(H)$$

$$=\left[Dv^{d}+(1-D)v^{nd}(X_r,H,T,c)\right]N$$

(18) $$=\left[v^{nd}(X_r,H,T,c)-\underbrace{D\left(v^{nd}(X_r,H,T,c)-\delta F\right)}_{\substack{\text{loss due to a default of } D \text{ percent of the}\\ \text{debtors within the risk horizon}}}\right]N\,.$$

If we assume that the portfolio of corporate coupon bonds is sufficiently large so that the fraction D of defaulting debtors given a certain realization of the risk factors $Z=z$ and $X_r=x_r$ is adequately approximated by the conditional default probability $q(z,x_r)$, we get with the law of iterated expectations for the probability distribution of the credit portfolio value $\Pi(H)$ at the risk horizon:

$$P\left(\Pi(H)\leq\pi\right)$$

$$=P\left(\left[v^{nd}(X_r,H,T,c)-D\left(v^{nd}(X_r,H,T,c)-\delta F\right)\right]N\leq\pi\right)$$

$$=E^{P}\left[P\left(\left[v^{nd}(X_r,H,T,c)-D\left(v^{nd}(X_r,H,T,c)-\delta F\right)\right]N\leq\pi\middle|Z,X_r\right)\right]$$

(19) $$=E^{P}\left[P\left(\left[v^{nd}(X_r,H,T,c)-q(Z,X_r)\left(v^{nd}(X_r,H,T,c)-\delta F\right)\right]N\leq\pi\middle|Z,X_r\right)\right].$$

For a detailed derivation of this probability distribution see the appendix.

III. Numerical Example

In this section a numerical example of the model presented in the previous section II shall demonstrate the effect of a variation of one of the parameters on the credit portfolios' standard deviation and Value-at-Risk (VaR) at the risk horizon. The VaR is defined as the difference between the expected credit portfolio value at the risk horizon and the respective percentile of the credit portfolio distribution.

The parameters of the Ornstein-Uhlenbeck process (2) modeling the short-term risk-free interest rate are from *Lehrbass* (1997), who estimated these parameters using the DEM-Libor overnight rates within the period July 31, 1991 to May 31, 1995. The market price of interest rate risk λ is computed as the average of the values calculated by *Lehrbass* (1997). The unconditional default probability in the base case is chosen as $0,7\%$. In the second consultative paper of the *Basel Committee on Banking Supervision* (2001a,b) this is the annual default probability to which a IRB-benchmark risk weight of 100% corresponds. According to Moody's all-corporate average rating transition matrix over the period 1980-2000 a debtor with a default probability of $0,7\%$ would have a rating between Baa ($0,16\%$) and Ba ($1,32\%$).[9] The credit spreads used for the following calculations are the average of the quadratic interpolated spreads which *Kiesel*, *Perraudin* and *Taylor* (1999) estimated for US industrials with a rating corresponding to a rating of Baa and Ba.[10] The recovery rate equals Moody's average recovery rate of senior unsecured bonds during 1981 to 1999.[11] In the base case, the value of the correlation of asset returns is chosen as 20%, which corresponds to the assumed value in the IRB-approach of Basel's second consultative paper. The parameter ω_2, which determines the correlation between the firm's asset returns and the term structure of interest rates, is set to $-\sqrt{0,1}$ implying a negative correlation between asset returns and interest rates. The coupon $c = 9,223\%$ is chosen in order to guarantee that a coupon bond with a time to maturity of $T = 3$ years equals its par value at $t = 0$.

Table 1 shows that the standard deviation of $\Pi(H)$ and the VaR values corresponding to different confidence levels become larger with the asset return correlation increasing. The values in the columns with the heading “without IR” are calculated by using the forward rates observed at $t = 0$ for revaluing the coupon bonds at the risk horizon. This is the proceeding applied for example in the credit portfolio model CreditMetrics™. In contrast to this, in the columns “with IR” the coupon bonds are revalued on the basis of the realized spot rates $R(X_r, H, t_m)$ for the different times to maturity t_m, which are of course unkown at $t = 0$. Because of the second stochastic factor which influences the coupon bond’s value at the risk horizon the standard deviations and VaR values are always higher in the case “with IR” than in the case “without IR”. As table 1 shows the differences between the risk measures with and without considering interest rate risk decrease with the correlation and the confidence level increasing. This demonstrates that credit risk becomes the dominating risk factor with the asset returns correlation increasing and/or going further to the “bad” end of the credit portfolio distribution at $t = H$. These findings are consistent with those of *Kiesel*, *Perraudin* and *Taylor* (1999) and *Grundke* (2002).

- insert table 1 about here -

Table 2 reports the standard deviations and VaR values for different values of the asset return’s sensitivity ω_2 to the interest rate factor X_r with the total correlation ρ of the asset returns held constant. As it can be seen, there is a decrease of the standard deviation and the VaR values with ω_2 increasing. This decrease can be explained by the changing covariance between the two summands in (A2) (given the realization of Z) with ω_2 rising from $-\sqrt{0{,}15}$ to $\sqrt{0{,}15}$.

- insert table 2 about here -

The values of table 3 show that, as expected, with the credit quality of the portfolio decreasing the standard deviation and the VaR values increase. Furthermore, it can be observed that the difference between the risk measures with and without integrated interest rate risk decreases with the default probability increasing. This effect can be explained by the shrinking relative importance of the interest rate risk, compared to the default risk, when the credit quality of the portfolio worsens. These findings are also consistent with those of *Kiesel*, *Perraudin* and *Taylor* (1999) and *Grundke* (2002).

- insert table 3 about here -

Table 4 shows the influence of changing the time to maturity of the coupon bonds on the standard deviation and the VaR values. Within the credit portfolio model presented in the previous section the standard deviation and the VaR values slightly decrease with the time to maturity increasing in the case that interest rate risk is ignored during the revaluation process.[12] With considering interest rate risk the standard deviation and the VaR values at first slightly increase and then decrease. The difference between the risk measures with and without considering interest rate risk exhibits a small increase with the time to maturity rising. *Kiesel*, *Perraudin* and *Taylor* (1999) find that the VaR values monotonously increase with the time to maturity increasing, and that the sensitivity of the VaR values to the time to maturity is much more significant. In correspondence with the findings of this paper they also observe that the difference between the VaR values with and without considering interest rate risk is higher the longer the time to maturity, but again this effect in much more pronounced in their model set-up.

- insert tables 4 about here -

Finally, in order to be able to assess more properly the magnitude of the interest rate risk within this numerical example, table 5 reports the influence of changing the time to maturity of the coupon bonds on the standard deviation and the VaR values in the (default) risk-free case. There is a slight increase in both risk measures for longer times to maturity. Together with the decreasing values of the standard deviation and the VaR in the case that interest rate risk is ignored during the revaluation process (see table 4, columns "without IR") this explains the peak of the risk measure values as a function of the time to maturity in the columns "with IR" of table 4. Furthermore, comparing the values in table 4 (columns "with IR") and table 5, it can be noticed that the difference between the risk measures in the risk-free case and in the default risky case increases the higher the confidence level. This again demonstrates the growing relative importance of credit risk the further we look at the "bad" end of the credit portfolio distribution.

- insert tables 5 about here -

IV. Summary and Conclusions

A typical shortcoming of most current credit portfolio models is the lack of a stochastic modeling of risk factors, such as interest rates or credit spreads, during the revaluation process at the risk horizon. Within the simple credit risk model of the IRB-approach of Basel II with incorporated correlated interest rate risk the effect which results form neglecting the stochastic nature of market risk factors is shown for an infinitely large, homogenous portfolio of defaultable coupon bonds. The consequence of ignoring interest rate risk can be that not enough capital is allocated as a protection against an unexpected deterioration of the portfolio's value. The lower the correlation of firms' asset returns, the lower the unconditional default probability and the longer the bonds' time to maturity, the higher is the difference between the standard deviation and the VaR values with and without considering interest rate risk during the revaluation process at the risk horizon.

The analysis conducted in this paper offers several possibilities for extensions. One extension would be to alter the composition of the credit portfolio, for example, by changing the type of financial instrument (e.g. interest rate options). Another direction consists in improving the credit portfolio model, in the sense of making it more realistic, for example, by modelling stochastic credit spreads which depend on the realized asset returns at the risk horizon.

Appendix:

Derivation of the probability distribution of the credit portfolio value Π(*H*) at the risk horizon

Using the assumption of a sufficiently large credit portfolio and the law of iterated expectations we get for the probability distribution of the credit portfolio value $\Pi(H)$ at the risk horizon:

$$P\left(\Pi(H) \leq \pi\right)$$

$$= E^P\left[P\left(\left[v^{nd}(X_r, H, T, c) - q(Z, X_r)\left(v^{nd}(X_r, H, T, c) - \delta F\right)\right]N \leq \pi \middle| Z, X_r\right)\right]$$

(A1) $$= E^P\left[P\left(\left[v^{nd}(X_r, H, T, c) - q(Z, X_r)\left(v^{nd}(X_r, H, T, c) - \delta F\right)\right]N \leq \pi \middle| X_r\right)\right].$$

Inserting the definition (14) of $q(Z, X_r)$ into (A1) and solving the inequation in the probability term for Z yields:

(A2) $$\left[v^{nd}(X_r, H, T, c) - \Phi\left(\frac{\alpha - \omega_1 Z - \omega_2 X_r}{\sqrt{1-\rho}}\right)\left(v^{nd}(X_r, H, T, c) - \delta F\right)\right]N \leq \pi$$

$$\Leftrightarrow -\Phi\left(\frac{\alpha - \omega_1 Z - \omega_2 X_r}{\sqrt{1-\rho}}\right)\left(v^{nd}(X_r, H, T, c) - \delta F\right) \leq \frac{\pi}{N} - v^{nd}(X_r, H, T, c)$$

(A3) $$\Leftrightarrow \Phi\left(\frac{\alpha - \omega_1 Z - \omega_2 X_r}{\sqrt{1-\rho}}\right)\left(v^{nd}(X_r, H, T, c) - \delta F\right) \geq v^{nd}(X_r, H, T, c) - \frac{\pi}{N}.$$

Now, we have to differ between the two cases that the second factor on the left-hand side is either negative or positive. This factor can be negative if the risk-free interest rates have become very large until the risk horizon so that the future coupon and face value payments are discounted very much and, hence, their value at the risk horizon is less than a fraction δ of the face value paid immediately at the risk horizon.[13]

I. $v^{nd}(X_r, H, T, c) - \delta F < 0$

Then, we get for (A3):

$$\text{(A4)} \quad \Leftrightarrow \underbrace{\Phi\left(\frac{\alpha - \omega_1 Z - \omega_2 X_r}{\sqrt{1-\rho}}\right)}_{\in[0,1]} \le \frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)}.$$

We further have to differentiate between the two cases that the numerator on the right-hand side is either negative or non-negative:

I. (i) $v^{nd}(X_r, H, T, c) - \frac{\pi}{N} < 0$

$$\Rightarrow \frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)} > 0$$

$\Rightarrow$ If $\frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)} > 1$, then the event (A2) has probability one;

$\Rightarrow$ If $0 < \frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)} \le 1$, then (A4) must be transformed as follows:

$$\Phi\left(\frac{\alpha - \omega_1 Z - \omega_2 X_r}{\sqrt{1-\rho}}\right) \le \frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)}$$

$$\Leftrightarrow \frac{\alpha - \omega_1 Z - \omega_2 X_r}{\sqrt{1-\rho}} \le \Phi^{-1}\left(\frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)}\right)$$

$$\text{(A5)} \quad \underset{\omega_1 > 0}{\Leftrightarrow} Z \ge \left(-\frac{1}{\omega_1}\right)\left(\sqrt{1-\rho}\,\Phi^{-1}\left(\frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)}\right) - \alpha + \omega_2 X_r\right).$$

Inserting (A5) in (A1) and using $Z \sim N(0,1)$ yields:

$$P\left(\Pi(H) \le \pi\right)$$

$$= E^P\left[\Phi\left(\left(\frac{1}{\omega_1}\right)\left(\sqrt{1-\rho}\,\Phi^{-1}\left(\frac{v^{nd}(X_r,H,T,c)-\frac{\pi}{N}}{\left(v^{nd}(X_r,H,T,c)-\delta F\right)}\right)-\alpha+\omega_2 X_r\right)\right)\right]$$

(A6) $$= \int_{-\infty}^{+\infty}\Phi\left(\left(\frac{1}{\omega_1}\right)\left(\sqrt{1-\rho}\,\Phi^{-1}\left(\frac{v^{nd}(x_r,H,T,c)-\frac{\pi}{N}}{\left(v^{nd}(x_r,H,T,c)-\delta F\right)}\right)-\alpha+\omega_2 x_r\right)\right)\phi(x_r)dx_r$$

with $\phi(x_r)$ denoting the density function of the standard normal distribution. The integral (A6) must be solved numerically.

I. (ii) $v^{nd}(X_r,H,T,c)-\frac{\pi}{N} \ge 0$

$$\Rightarrow \frac{v^{nd}(X_r,H,T,c)-\frac{\pi}{N}}{\left(v^{nd}(X_r,H,T,c)-\delta F\right)} \le 0$$

$\Rightarrow$ The event (A2) has probability zero.

II. $v^{nd}(X_r,H,T,c)-\delta F > 0$

Then, we get for (A3):

(A7) $$\Leftrightarrow \underbrace{\Phi\left(\frac{\alpha-\omega_1 Z-\omega_2 X_r}{\sqrt{1-\rho}}\right)}_{\in[0,1]} \ge \frac{v^{nd}(X_r,H,T,c)-\frac{\pi}{N}}{\left(v^{nd}(X_r,H,T,c)-\delta F\right)}.$$

II. (i) $v^{nd}(X_r,H,T,c)-\frac{\pi}{N} < 0$

$$\Rightarrow \frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)} < 0$$

$\Rightarrow$ The event (A2) has probability one.

II. (ii) $v^{nd}(X_r, H, T, c) - \frac{\pi}{N} \geq 0$

$$\Rightarrow \frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)} \geq 0$$

$\Rightarrow$ If $\frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)} > 1$, then the event (A2) has probability zero;

$\Rightarrow$ If $0 \leq \frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)} \leq 1$, then (A7) must be transformed as follows:

$$\Phi\left(\frac{\alpha - \omega_1 Z - \omega_2 X_r}{\sqrt{1-\rho}}\right) \geq \frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)}$$

(A8) $$\underset{\omega_1 > 0}{\Leftrightarrow} Z \leq \left(-\frac{1}{\omega_1}\right)\left(\sqrt{1-\rho}\,\Phi^{-1}\left(\frac{v^{nd}(X_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(X_r, H, T, c) - \delta F\right)}\right) - \alpha + \omega_2 X_r\right).$$

This yields:

$$P\left(\Pi(H) \leq \pi\right)$$

(A9) $$= \int_{-\infty}^{+\infty} \Phi\left(\left(-\frac{1}{\omega_1}\right)\left(\sqrt{1-\rho}\,\Phi^{-1}\left(\frac{v^{nd}(x_r, H, T, c) - \frac{\pi}{N}}{\left(v^{nd}(x_r, H, T, c) - \delta F\right)}\right) - \alpha + \omega_2 x_r\right)\right) \phi(x_r) dx_r .$$

TABLES:

Table 1: VaR of a portfolio of coupon bonds for different values of the asset returns' correlation ρ

ρ			$\rho = 0{,}15$		$\rho = 0{,}2$		$\rho = 0{,}3$		$\rho = 0{,}4$	
			without IR	with IR	without IR	with IR	without IR	with IR	without IR	with IR
current value of the credit portfolio $\Pi(0)$			1000,01							
expected value of the credit portfolio at the risk horizon $E^{P}_{r(0)}[\Pi(H)]$			1080,64	1091,90	1080,64	1091,90	1080,64	1091,90	1080,64	1091,90
standard deviation of the credit portfolio value at the risk horizon $\sigma(\Pi(H))$			5,40	17,59	6,69	18,02	9,39	19,16	12,34	20,75
		+	0,50%	1,61%	0,62%	1,65%	0,87%	1,75%	1,14%	1,90%
		++		30,70%		37,13%		49,01%		59,47%
VaR $:= E^{P}_{r(0)}[\Pi(H)] - 100(1-p)$th percentile	p									
	0,95		9,84	30,48	11,54	30,98	14,00	31,64	15,30	31,74
		*	0,91%	2,79%	1,07%	2,84%	1,30%	2,90%	1,42%	2,91%
		**		32,28%		37,25%		44,25%		48,20%
	0,99		22,20	49,67	28,44	53,18	41,10	61,68	54,22	71,90
		*	2,05%	4,55%	2,63%	4,87%	3,80%	5,65%	5,02%	6,58%
		**		44,69%		53,48%		66,63%		75,41%
	0.999		45,20	79,24	61,74	91,34	99,45	122,89	144,05	163,56
		*	4,18%	7,26%	5,71%	8,37%	9,20%	11,25%	13,33%	14,98%
		**		57,04%		67,59%		80,93%		88,07%

without IR: without considering interest rate risk for the bonds' revaluation at the risk horizon
with IR: with considering interest rate risk for the bonds' revaluation at the risk horizon

$^{+}$: $100\% \cdot \sigma(\Pi(H))/E^{P}_{r(0)}[\Pi(H)]$; $^{++}$: $100\% \cdot \sigma(\Pi(H))^{\text{without IR}}/\sigma(\Pi(H))^{\text{with IR}}$; * : $100\% \cdot \mathrm{VaR}/E^{P}_{r(0)}[\Pi(H)]$;

** : $100\% \cdot \mathrm{VaR}^{\text{without IR}}/\mathrm{VaR}^{\text{with IR}}$

Parameters:

$N = 1000$, $F = 1$, $T = 3$, $H = 1$, $t_m \in \{1,2,3\}$, $c = 0{,}09223$, $\omega_2 = -\sqrt{0{,}1}$, $\alpha = -2{,}4573$, $\delta = 0{,}511$, $S(0,1,2) = 0{,}01196$, $S(0,2,3) = 0{,}01263$, $\kappa = 1{,}169$, $\theta = 0{,}061$, $\sigma = 0{,}029$, $\lambda = 0{,}88$, $r(0) = 0{,}061$

Table 2: VaR for different values of the asset returns' sensitivity ω_2 to interest rate risk

ω_2, ω_1			$\omega_2 = -\sqrt{0,15}$, $\omega_1 = \sqrt{0,05}$	$\omega_2 = -\sqrt{0,05}$, $\omega_1 = \sqrt{0,15}$	$\omega_2 = 0$, $\omega_1 = \sqrt{0,2}$	$\omega_2 = \sqrt{0,05}$, $\omega_1 = \sqrt{0,15}$	$\omega_2 = \sqrt{0,15}$, $\omega_1 = \sqrt{0,05}$
$\Pi(0)$			1000,01				
$E^P_{r(0)}[\Pi(H)]$			1091,92	1091,88	1091,82	1091,76	1091,72
$\sigma(\Pi(H))$			18,57	17,26	15,22	12,78	10,58
		+	1,70%	1,58%	1,39%	1,17%	0,97%
VaR $:= E^P_{r(0)}[\Pi(H)]$ $-100(1-p)$th percentile	p						
	0,95		32,43	29,12	24,94	20,71	19,31
		*	2,97%	2,67%	2,28%	1,90%	1,77%
	0,99		56,27	49,04	38,82	31,01	28,27
		*	5,15%	4,49%	3,56%	2,84%	2,59%
	0.999		96,03	85,00	68,25	49,20	41,42
		*	8,79%	7,78%	6,25%	4,51%	3,79%

+, * : See table 1
Parameters: See table 1

Table 3: VaR for different unconditional default probabilities q of the debtors

q			$q=0,007$		$q=0,02$		$q=0,05$	
			without IR	with IR	without IR	with IR	without IR	with IR
$\Pi(0)$			1000,01					
$E^P_{r(0)}[\Pi(H)]$			1080,64	1091,90	1073,18	1084,42	1055,98	1067,12
$\sigma(\Pi(H))$			6,69	18,02	15,18	25,36	30,06	39,47
		+	0,62%	1,65%	1,41%	2,34%	2,85%	3,70%
		++		37,14%		59,86%		76,15%
VaR $:= E^P_{r(0)}[\Pi(H)] - 100(1-p)\text{th percentile}$	p							
	0,95		11,54	30,98	28,84	46,36	60,04	76,09
		*	1,07%	2,84%	2,69%	4,28%	5,69%	7,13%
		**		37,26%		62,22%		78,91%
	0,99		28,44	53,18	62,31	84,76	114,48	134,73
		*	2,63%	4,87%	5,81%	7,82%	10,84%	12,63%
		**		53,48%		73,51%		84,97%
	0.999		61,75	91,35	118,36	144,86	191,84	214,81
		*	5,71%	8,37%	11,03%	13,36%	18,17%	20,13%
		**		67,59%		81,71%		89,30%

without IR, with IR: See table 1

$^{+}$, $^{++}$, *, **: See table 1

Parameters: $\rho = 0,2$, all other values equal to table 1

Table 4: VaR for different times to maturity *T* of the defaultable coupon bonds

T			$T=3$		$T=6$		$T=9$	
			without IR	with IR	without IR	with IR	without IR	with IR
$\Pi(0)$			1000,01		982,77		965,31	
$E^P_{r(0)}[\Pi(H)]$			1080,63	1091,90	1062,07	1074,08	1043,26	1055,05
$\sigma(\Pi(H))$			6,69	18,02	6,47	18,71	6,25	18,28
		+	0,62%	1,65%	0,61%	1,74%	0,60%	1,73%
		++		37,14%		34,60%		34,21%
	p							
VaR $:= E^P_{r(0)}[\Pi(H)] - 100(1-p)$th percentile	0,95		11,54	30,98	11,17	32,06	10,79	31,32
		*	1,07%	2,84%	1,05%	2,98%	1,03%	2,97%
		**		37,26%		34,83%		34,45%
	0,99		28,44	53,18	27,51	54,10	26,57	52,71
		*	2,63%	4,87%	2,59%	5,04%	2,55%	5,00%
		**		53,48%		50,86%		50,42%
	0,999		61,75	91,35	59,74	91,45	57,70	88,86
		*	5,71%	8,37%	5,62%	8,51%	5,53%	8,42%
		**		67,59%		65,33%		64,93%

without IR, with IR: See table 1
+, ++, *, **: See table 1
Parameters: $T \in \{3,6,9\}$, $t_m \in \{1,2,\ldots,9\}$, $\rho = 0,2$, all other values equal to table 1

Table 5: VaR for different times to maturity *T* of the risk-free coupon bonds

T			$T=3$	$T=6$	$T=9$
$\Pi(0)$			1033,46	1046,45	1056,28
$E^P_{r(0)}[\Pi(H)]$			1119,81	1134,92	1145,62
$\sigma(\Pi(H))$			14,03	15,47	15,67
		+	1,25%	1,36%	1,37%
VaR $:= E^P_{r(0)}[\Pi(H)]$ $-100(1-p)$th percentile	p				
	0,95		22,85	25,18	25,49
		*	2,04%	2,22%	2,22%
	0,99		32,14	35,39	35,83
		*	2,87%	3,12%	3,13%
	0,999		42,68	46,98	47,57
		*	3,81%	4,14%	4,15%

+, * : See table 1

Parameters: $N=1000$, $F=1$, $T\in\{3,6,9\}$, $H=1$, $t_m\in\{1,2,\ldots,9\}$, $c=0,09223$, $\kappa=1,169$, $\theta=0,061$, $\sigma=0,029$, $\lambda=0,88$, $r(0)=0,061$

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ENDNOTES

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1 Making us of Monte Carlo-simulations *Kiesel*, *Perraudin* and *Taylor* (1999) investigate within the framework of JP Morgan's CreditMetrics™ (see *Gupton*, *Finger* and *Bhatia* (1997)) the consequences which result from adding credit spread risk to the model for a portfolio of defaultable zero coupon bonds. The risk-free interest rates are assumed to be constant. *Kijima* and *Muromachi* (2000) integrate interest rate risk into an intensity-based credit portfolio model. The risk-free short-term interest rate and the intensity of the default time of each debtor are modeled as correlated extended Vasicek processes originally proposed by *Hull* and *White* (1990). *Kijima* and *Muromachi* (2000) also use Monte-Carlo-methods and deal with a portfolio of defaultable zero coupon bonds. *Barth* (2000) computes by Monte Carlo-simulations various worst case risk measures for a portfolio of interest rate swaps with counterparty risk. The risk-free short-term interest rate is modeled as a square-root process used by *Cox*, *Ingersoll* and *Ross* (1985) and is part of the counterparty specific default time intensity so that interest and credit risk are correlated. The most extensive study with regard to the number of simulated risk factors is from *Barnhill* and *Maxwell* (2002). They simulate the risk-free term structure, the credit spreads of each rating class, a foreign exchange rate and equity market indices, which are all assumed to be correlated. The individual firm's return on equity is deduced from the return on the market index applying the CAPM model. These individual equity returns are then used to compute the firm's debt ratio, which are mapped into credit ratings. Knowing the firm's credit rating at the risk horizon the appropriate (simulated) risk-adjusted term structure of interest rates can be used for discounting the future cash flows of the portfolio of coupon bonds.

A first attempt to integrate stochastic market risk factors into the revaluation process at the risk horizon can be found in the commercial credit portfolio model Portfolio Credit Risk Engine by Algorithmics (see *Iscoe*, *Kreinin* and *Rosen* (1999)).

[2] See *Basel Committee on Banking Supervision* (2001a,b; 2003).

[3] A one-factor version of this model, without integrated interest rate risk, is used for example by *Finger* (1999), *Gordy* (2000), *Schönbucher* (2001) or *Vasicek* (2002).

[4] See *de Munnik* (1996, p. 70); *Vasicek* (1977, p. 185).

[5] For example the ‘extended’ Vasicek-model of *Hull* and *White* (1990).

[6] See *de Munnik* (1996, p. 71); *Vasicek* (1977, pp. 185).

[7] See for example *Billingsley* (1995, pp. 282).

[8] In contrast to the model presented here, CreditPortfolioView™ also ignores interest rate risk when revaluing for example corporate bonds at the risk horizon and uses instead the risk-adjusted forward rates of each rating class.

[9] See *Moody’s Investors Service* (2001, p. 44).

[10] See *Kiesel*, *Perraudin* and *Taylor* (1999, pp. 8, p. 25), who use daily Bloomberg spread data covering the period April 1991 to November 1998. The spreads are calculated as the difference between the yields of notional zero coupon bonds with different ratings and times to maturity of 2, 5 and 10 years issued by US industrials and the yields of US Treasury strips of the same time to maturity.

[11] See *Moody’s Investors Service* (2001, p. 26).

[12] Ignoring interest rate risk and choosing the coupon for each time to maturity individually, so that the value of the coupon bonds in $t = 0$ equals its par value, the risk measures remain constant when varying the time to maturity.

[13] For the parameters chosen in the numerical example in the following section III this can only happen with extreme low probability.