Firm Valuation in a Continuous-Time SDF Framework*

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Abstract

This paper proposes a unifying SDF framework to firm valuation in continuous time that nests all existing firm valuation models. It generalizes the fundamental asset pricing equation by introducing controlled state variable, discount factor, and cash flow processes. The generalized SDF framework of asset pricing constitutes the basis of all firm valuation models. The SDF approach to firm valuation displays several advantages: it can handle models from the contingent claims, real options, and asset pricing literature in a consistent manner; it integrates no-arbitrage and general equilibrium models; it allows for an easy formalization of qualitative notions as e.g. control premia; and it bridges the gap between firm valuation models aiming at the determination of market values and those aiming at the computation of subjective firm values for an individual investor. Furthermore, the SDF framework highlights that firm valuation models can differ in only 6 dimensions: state variables, SDF derivation, SDF specification, cash flow process, set of feasible control laws, and applicable boundary conditions. The existing continuous-time firm valuation models of Gordon [1962], Brennan and Schwartz [1982a,b, 1984], Bakshi and Chen [2001], and Schwartz and Moon [2000, 2001] are derived as special cases of the generalized SDF framework and related to each other.

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1 Introduction

Due to its central role in finance, the valuation of companies has been addressed by various strands of the financial literature. Essentially, the existing firm valuation models can be categorized in four classes. They either belong to the classical firm valuation, the contingent claims, the real options, or the asset pricing literature. Their differing backgrounds imply that the models do not share a common framework. Instead, they appear to be distinct approaches. This impression is reinforced by two facts. First, authors from one strand of the literature hardly relate their work to that done in other areas.[1] Second, there does not exist any work in the financial literature that recognizes the links between the different models and relates them to each other.[2] Yet, despite their roots in different areas of financial research, the models are closely related.

In this paper, we therefore bring together the firm valuation models put forward in various strands of the financial literature. We propose a unifying SDF framework of firm valuation in continuous time and show that the existing models are nested within this framework. In deriving the existing models as special cases of the general SDF framework, we restrict our attention to those models that address the valuation of the firm as a whole and not the valuation of specific corporate securities. This implies that we do not explicitly integrate the numerous contingent claims models in the SDF framework as these focus on the valuation of specific corporate securities. Given the detailed exposition in subsequent sections, the derivation of all these models within the SDF framework is straightforward but would probably fill several papers of its own. Furthermore, we are only concerned with those approaches that are applicable to a wide class of companies. In the real options literature, there exist several firm valuation models intended for the valuation of very specific types of firms. Brennan and Schwartz [1985] address e.g. the valuation of natural resource firms while Kellogg and Charnes [2000] are concerned with the valuation of biotechnology companies. These approaches are also not explicitly derived within the general SDF framework. The reasons are the same as for the contingent claims models. However, it will also become clear how the existing real options models could be developed within the general framework.

The remaining paper is structured as follows. In the next section, we review the literature on firm valuation in continuous time. In section 3, we generalize the continuous-time SDF framework of asset pricing by introducing controlled state variable, discount factor, and cash flow processes. We argue that the generalized framework constitutes the basis of all firm valuation approaches and highlight its advantages in the context of firm valuation. Building on the generalized SDF framework, we derive several differential equations that are satisfied by all asset pricing and firm valuation models in continuous time. Section 4 develops the existing no-arbitrage models of firm valuation within the

[1] Bakshi and Chen [2001] e.g. do not cite any general equilibrium model of firm valuation. Yet, their approach is closely related to the general equilibrium models in the asset pricing literature as is shown in sections 4 and 5.

[2] An exception is the paper of Pastor and Veronesi [2002]. Although they do not relate the models to each other, they at least cite most works from the different strands of literature.

generalized SDF framework. In these models, the stochastic discount factor is given exogenously. Specifically, we deduce the models of Gordon [1962] and Bakshi and Chen [2001] as special cases of the basic framework. In section 5, we turn to general equilibrium models of firm valuation. In this part, the models of Brennan and Schwartz [1982a,b, 1984] and Schwartz and Moon [2000, 2001] are derived as special cases of the generalized SDF framework. The stochastic discount factor in these models has to be determined endogenously. We show that the stochastic discount factor implied by these models is consistent with a particular specification of the production economy of Cox, Ingersoll, and Ross [1985b]. We further relate the models to each other and demonstrate that some models are nested within others. Section 6 summarizes and concludes.

2 Related Literature

The valuation of companies has a long standing tradition in the finance literature with the earliest works dating back more than 60 years. One of the first contributions is the book of Williams [1938]. Originally written as a Ph.D. thesis at Harvard University, the book has become a classic that is still in print without any revisions. Building on the net present value rule as suggested by Fisher [1930], Williams [1938] develops the first *Dividend Discount Model* to the valuation of companies. He considers three special cases, namely that the firm's dividends remain constant, grow forever at a constant rate, or follow an S-shaped curve over time. For each of these cases, he derives an analytical formula for the company's value. William's work stands out from the finance theory on the valuation of firms through the early 1950s as most works were not based on solid theoretical grounds but consisted largely of ad hoc propositions.

2.1 Classical Firm Valuation Literature

The first solid theoretical article concerned with the valuation of companies is the work of Modigliani and Miller [1958]. Together with the paper of Markowitz [1952], the work of Modigliani and Miller [1958] is today considered as the beginning of modern finance. In their paper, Modigliani and Miller [1958] analyze the impact of the firm's capital structure on the value of the firm. Building on a no-arbitrage argument, they demonstrate that, ignoring taxes and transaction costs and given the firm's investment policy, the firm's value is independent of its capital structure. Without any doubt, this irrelevance proposition is one of the most important findings not only of the early firm valuation but of the finance literature in general. In the same article and especially in a later correction, Modigliani and Miller [1963] also analyze the capital structure choice in a world with taxes.

Three years after the capital structure irrelevance proposition, Miller and Modigliani [1961] published another path-breaking article. Relying again on a no-arbitrage argument, they prove the irrelevance of the firm's dividend policy for valuation in perfect capital markets. Furthermore, they show that both the *Dividend Discount Model* and the *Discounted Cash Flow Model* of firm valuation directly follow from the valuation principle in perfect capital markets and are equivalent to each other. Until today, these two approaches have remained the workhorses of the firm valuation literature.

Gordon and Shapiro [1956] and Gordon [1962] propose the first parametric model of firm valuation. Assuming that the firm's earnings grow forever at a constant rate, they derive a simple analytical formula for the firm value, the famous Gordon Growth Model.[3] Together, the articles of Modigliani and Miller [1958, 1961, 1963] and the book of Gordon [1962] constitute the most important contributions of the classical firm valuation literature.

2.2 Contingent Claims Literature

The next significant advances in firm valuation emerged from the contingent claims literature. It was the first strand of literature to systematically apply continuous-time methods to firm valuation problems. In 1973, Black and Scholes [1973] and Merton [1973b] published their seminal articles on the pricing of European call options. Already in these early works, Black and Scholes [1973] and Merton [1973b] emphasize the applicability of their methodology to the valuation of corporate claims. The analogy between options and corporate securities is most obvious if the firm's only outstanding claims are a zero coupon bond and common stock. In this case, the shareholders' position is equivalent to a European call option on the firm's assets with the strike price given by the nominal amount of the zero coupon bond. This insight also constitutes the basis of the first structural credit risk model for the pricing of corporate debt by Merton [1974]. In the aftermath, contingent claims analysis has been applied for pricing all kinds of corporate securities, ranging from stock over preferred stock and corporate coupon bonds to convertible securities. Important articles in these areas are Emanuel [1983], Black and Cox [1976], Geske [1977], Brennan and Schwartz [1977], and Ingersoll [1977].[4]

Although contingent claims approaches have made substantial contributions to the firm valuation literature, most papers of this literature strand share a common shortcoming. They assume the firm value to follow an exogenously specified stochastic process. The firm's outstanding securities are then considered as derivatives written on the underlying firm value process. This setup implies that the models miss out the valuation of the firm as a whole and the determination of the firm value process. The motivation behind this modeling approach is obvious. First, the assumptions assure the perfect analogy between option pricing and the valuation of corporate claims and thus facilitate the transfer of option pricing results to the valuation of corporate securities. Second, by employing only financial variables, the valuation of the firm's various claims can be based on pure arbitrage arguments. Consequently, contingent claims approaches are highly suitable for the valuation of a company's individ-

[3]The original, and more general, formula stems from Williams [1938, equation (17a)], Gordon and Shapiro [1956, equation (7)] and Gordon [1962, equation (4.6)] only rediscovered and specialized it. The formula is also developed in Miller and Modigliani [1961, equation (23)].

[4]For an overview of this literature, see Mason and Merton [1985] and Merton [1990, pp. 423–427].

ual securities *given* the firm's value, but they are of little help in valuing the company in the first place.

2.3 Real Options Literature

Since the middle of the 1980s, contingent claims approaches have been extended to the analysis and valuation of companies' operating options, as e.g. the option to postpone an investment. These applications constitute what is known as the real options literature.[5] This literature has also examined firm valuation questions in continuous time. The first and probably most important contribution in this area is the article of Brennan and Schwartz [1985]. They value a natural resource mine including the options to temporarily close, reopen, and shutdown the mine. Most other papers in the real options literature have focused on the valuation of natural resource companies as well.[6] This focus rests on the easy modeling of the underlying risk factors for these firms. Namely, it seems natural to use the price of the commodity as the underlying risk factor. Since commodity products or commodity futures are exchange-traded, one can again apply pure arbitrage arguments to solve the valuation problem.

Some real options papers have addressed the valuation of other companies besides natural resource firms. Ottoo [1998] and Kellogg and Charnes [2000] derive e.g. models for the valuation of biotechnology firms. Willner [1995] develops a real options approach targeted at the valuation of start-up companies. However, the model has two drawbacks. First, it focuses on the valuation of the firm's growth options but neglects the valuation of the existing business. Second, the model is suitable only for a narrow class of firms. Willner [1995] presumes that the underlying investment projects follow a pure jump process. This assumption seems to be appropriate only for very R&D-intensive firms but not for the majority of companies.

2.4 Asset Pricing Literature

The fourth strand of literature on firm valuation in continuous time originated from the asset pricing literature. The majority of asset pricing models of firm valuation are based on the intertemporal general equilibrium asset pricing model of Cox, Ingersoll, and Ross [1985b]. In this paper, Cox, Ingersoll, and Ross [1985b] propose a continuous-time model of a production economy that integrates real and financial markets. The incorporation of real economic variables addresses a shortcoming of the contingent claims and real options approaches that are restricted to the modeling of financial variables. Building on assumptions about the evolution of the fundamental risk factors and the real production possibilities in the economy, Cox, Ingersoll, and Ross [1985b] derive a partial differential equation (PDE) that is satisfied by any security.[7] Consequently, the firm value must satisfy this equation as well.

[5] For reviews of the real options literature, see the books of Dixit and Pindyck [1994] and Trigeorgis [1996].

[6] See e.g. Paddock, Siegel, and Smith [1988] who value offshore petroleum leases and Morck, Schwartz, and Stangeland [1989] who assess the value of forestry resources.

[7] See Cox, Ingersoll, and Ross [1985b, p. 377].

On the basis of the model of Cox, Ingersoll, and Ross [1985b], Brennan and Schwartz [1982a,b, 1984] develop firm valuation models in continuous time. These models constitute the first continuous-time approaches to the valuation of the entire firm. Furthermore, their models include for the first time real economic variables thus going beyond the contingent claims and real options models that are limited to financial variables. Although the models of Brennan and Schwartz [1982a,b, 1984] differ in several dimensions, they share a common setup. Basically, the models are dividend-based approaches in the tradition of Gordon [1962]. The dividends are assumed to depend on the firm's cash flows, which are determined by the evolution of the firm's book value of assets and return on assets.[8]

The firm valuation models of Schwartz and Moon [2000, 2001] are also based on the general equilibrium model of Cox, Ingersoll, and Ross [1985b].[9] In contrast to the approaches of Brennan and Schwartz [1982a,b, 1984], the models of Schwartz and Moon [2000, 2001] are not dividend-based. Instead, they are cash flow-based comparable to the *Discounted Cash Flow* methodology. The methodologies of Schwartz and Moon [2000, 2001] are specifically targeted at the valuation of growth companies. Therefore, these models take the firm's revenue process as the main underlying risk factor and not the return on assets.[10]

While the models of Brennan and Schwartz [1982a,b, 1984] and Schwartz and Moon [2000, 2001] are general equilibrium models, the approach of Bakshi and Chen [2001] constitutes a no-arbitrage asset pricing model. Thus, the discount factor in Bakshi and Chen [2001] is specified exogenously whereas it is derived endogenously in the general equilibrium models.[11] The model of Bakshi and Chen [2001] is again dividend-based. Comparable to Gordon [1962] and Brennan and Schwartz [1982a,b, 1984], Bakshi and Chen [2001] model the underlying variables of the firm's dividend process. Specifically, they propose a stochastic process for the earnings evolution of the firm.[12]

Finally, there also exist numerous firm valuation models in discrete time that are rooted in the asset pricing literature. The most important models in discrete time are Campbell and Shiller [1987, 1988], Berk, Green, and Naik [1999], Lee, Myers, and Swaminathan [1999], Ang and Liu [2001], and Bekaert and Grenadier [2001]. However, as this paper focuses on continuous-time approaches, these models are not analyzed here in detail.

[8] For the models' detailed assumptions, see section 5.1.

[9] In their first paper, Schwartz and Moon [2000] classify their model as a real options approach. In the second paper, they still interpret it as a real options model although they use the JEL classification for asset pricing. See Schwartz and Moon [2001, p. 7]. The derivation of the model in section 5.2 clearly demonstrates that the model belongs to the asset pricing literature as it constitutes a special case of Cox, Ingersoll, and Ross [1985b].

[10] For a detailed description of the models, see section 5.2.

[11] For a detailed differentiation between no-arbitrage and general equilibrium models, see section 3.2.

[12] For the detailed assumptions, see section 4.2.

3 Firm Valuation in a SDF Framework

This section reviews and generalizes the SDF framework of asset pricing in continuous time. It starts with a description of several mathematical concepts that are useful in subsequent analyses. After a review and an economic interpretation of the standard SDF framework of asset pricing, we generalize the standard framework by introducing controlled state variable, discount factor, and cash flow processes. It is shown that the generalized framework is able to cope with various issues in asset pricing that cannot be handled in the original approach. Building on the insight that firm valuation is essentially part of asset pricing, it is argued that the generalized framework also constitutes a SDF approach to firm valuation. We then assess the implications and advantages of the generalized SDF framework of firm valuation. Finally, we derive several differential equations from the generalized SDF framework that are satisfied by all asset pricing and firm valuation models in continuous time.

3.1 Probabilistic Setup

Although our emphasis is on economic intuition and not on mathematical rigor, it is helpful to define some mathematical preliminaries. We thereby focus on those concepts that are essential for the analyses and models in subsequent sections.[13] The probabilistic structure of our economy is described by a probability space $(\Omega, \mathcal{F}, P)$. Here Ω denotes a set, $\mathcal{F}$ a σ-algebra of subsets of Ω, and P a probability measure on $\mathcal{F}$. The fixed time interval is given by $t \in [0, \infty)$. A family $\mathfrak{F} = \{\mathcal{F}_t : t \in [0, \infty)\}$ of sub-σ-algebras $\mathcal{F}_t$ of $\mathcal{F}$ is called a filtration. Thereby, it holds that $\mathcal{F}_t \subseteq \mathcal{F}_s$ for $t \leq s$.

The economic interpretation of this mathematical setup is straightforward. The set Ω contains the possible states of the world which are denoted by ω. A σ-algebra $\mathcal{F}$ of subsets of Ω can then be thought of as the set of identifiable events that can be assigned a probability. More specifically, $\mathcal{F}$ contains the events which investors are able to distinguish, i.e. it describes the information available to investors. The probability measure P assigns to any event B in $\mathcal{F}$ its probability $P(B)$. Throughout the paper, P is taken as the empirical probability measure, i.e. the probabilities assigned to specific events by P are the real probabilities. If not explicitly stated otherwise, all processes are specified under the measure P. Since we are dealing with continuous-time models, we must also describe how uncertainty is resolved or, stated differently, how information is revealed over time. This is exactly what is captured by the concept of a filtration $\mathfrak{F}$. $\mathcal{F}_t$ denotes the set of events that can be differentiated at time t. Thus, at time t one knows for sure whether a specific event in $\mathcal{F}_t$ has occurred or not. It is easiest to imagine $\mathcal{F}_t$ as the finest partition of Ω available at time t. The assumption $\mathcal{F}_t \subseteq \mathcal{F}_s$ for $t \leq s$ means that the partition $\mathcal{F}_s$ is finer than $\mathcal{F}_t$. This has two implications. First, it ensures that investors do not forget past information since the information at time t is contained in subsequent sub-σ-

[13] For a detailed exposition of the following mathematical concepts including their interpretation in economic and financial applications, see Huang and Litzenberger [1988], Neftci [2000], and Duffie [2001]. For a rigorous mathematical treatment, see Billingsley [1995].

algebras $\mathcal{F}_s$ for $s \geq t$. Second and more important, it implies that the partition and thus the information at time s is finer than at time t. Hence, the true state of the world is revealed bit by bit as time passes and the sub-σ-algebras become finer and finer. Based on the preceding explanations, it is intuitively clear why sub-σ-algebras are alternatively designated as information sets and a filtration is also called an information structure.

3.2 Continuous-Time SDF Framework

With the mathematical preliminaries behind us, we now turn to the SDF framework of asset pricing in continuous time. The fundamental asset pricing equation in continuous time is given by[14]

$$V(K(t), t) = E_t^P \left[\int_t^{\infty} \frac{\Lambda(K(s), s)}{\Lambda(K(t), t)} A(K(s), s) \, ds \right], \tag{1}$$

where $V(K(t), t)$ denotes the value of the asset at time t, $K(t)$ the k-dimensional vector of state variables, $E_t^P[\cdot]$ the expectation operator under the empirical probability measure P conditional on the information set at time t, $\Lambda(K(s), s)/\Lambda(K(t), t)$ the stochastic discount factor, and $A(K(s), s)$ the cash flow rate of the asset.[15] The discount factor $\Lambda(K, s)/\Lambda(K, t)$ and the cash flow rate $A(K, t)$ are both functions of the state variable vector $K(t)$. Any asset in a continuous-time economy must satisfy the fundamental asset pricing equation (1). Stated differently, this equation prices any available cash flow stream. According to (1), the value of any asset $V(K, t)$ is given by the expected value under the measure P of the integral over the asset's future cash flows deflated with the stochastic discount factor. The fundamental asset pricing equation can be derived and interpreted in two ways, either from a no-arbitrage or an equilibrium point of view.

We first adopt the no-arbitrage perspective. Therefore, we make a single assumption, namely that the capital market in the economy does not permit arbitrage profits, i.e. it is arbitrage-free. Then Harrison and Kreps [1979] prove that there exists a strictly positive stochastic discount factor that prices any asset in the economy.[16] This is a powerful result. It implies that the sole assumption of no arbitrage is sufficient to ensure that the price of any security in the economy can be represented by (1). Under the additional assumption that the capital market in the economy is complete, the stochastic discount

[14] See Cochrane and Saá-Requejo [2000, equation (21)] and Cochrane [2001, equation (1.28)].

[15] $\Lambda(K(s), s)/\Lambda(K(t), t)$ is the continuous-time equivalent to the standard stochastic discount factor terminology in a discrete-time setting. See Ingersoll [1987, p. 223], Cochrane and Saá-Requejo [2000, pp. 94–95], and Cochrane [2001, p. 30]. In order to simplify notation, the arguments of functional arguments are dropped hereafter. For example, we write $V(K, t)$ instead of $V(K(t), t)$.

[16] More specifically, Harrison and Kreps [1979] show that there exists a strictly positive stochastic discount factor iff the capital market is arbitrage-free. In their original article, Harrison and Kreps [1979] additionally impose the assumption that the capital market is frictionless. However, the theory can be extended to cope with market imperfections. See also Harrison and Pliska [1981].

factor is unique. In this case, the prices of all assets are uniquely determined by the above equation.

Economically, the stochastic discount factor implied by an arbitrage-free and complete capital market can be interpreted as the state prices in the economy normalized by the respective empirical state probabilities.[17] According to the fundamental asset pricing equation, today's price of any asset is then given by the expected value of the integral over the asset's future cash flows in each state weighted by the respective normalized state prices.[18] In an arbitrage-free and complete capital market, the state prices are unique. They are either directly observable or extractable from the prices of existing assets. If additionally the empirical probability density function is known, the stochastic discount factor can be derived explicitly. In this case, (1) can be directly applied to value any asset. In case the empirical probability density function is unknown, it is not possible to compute the stochastic discount factor explicitly. Nevertheless, the fundamental asset pricing equation can be used. Rearranging (1) leads to an alternative expression that prices any asset by weighting the asset's cash flows in each state by the respective state prices. Under the rearranged form of (1), the available state prices are sufficient for the valuation of any asset in the economy.

Having highlighted the derivation and interpretation of the fundamental asset pricing equation from a no-arbitrage point of view, we now adopt an equilibrium perspective. The equilibrium analysis of (1) starts with an agent's consumption and portfolio selection problem. Considering an individual agent implies a partial equilibrium approach towards the fundamental asset pricing equation.[19] Taking the agent to be the representative investor in the economy leads to a general equilibrium characterization of (1). Both approaches are formally identical but differ with respect to their interpretation. Here and throughout the paper, we adopt the general equilibrium perspective. The analysis of the consumption and portfolio selection problem starts with the specification of the representative investor's expected lifetime utility which is given by

$$U\left(\{c\}, K, t\right) = E_t^P \left[\int_t^\infty u\left(c, K, s\right)\, ds \right]. \tag{2}$$

Here $U\left(\{c\}, K, t\right)$ stands for the investor's expected lifetime utility, $\{c(t)\}$ for a consumption path starting at time t, and $u\left(c, K, s\right)$ for the investor's time and state dependent instantaneous utility function defined over the aggregate consumption rate $c(s)$.[20] The utility function $u\left(c, K, s\right)$ is assumed to be strictly

[17]See Cochrane [2001]. A state price is the price of an asset that pays \$1.00 if one particular state of the world occurs and nothing otherwise.

[18]The state prices are normalized by the empirical state probabilities as the probabilities are already captured by the expectation operator. Thus, the normalized state prices only reflect investors' marginal utility in different states of nature but not the empirical probabilities of the states.

[19]Duffie [2001] calls this the individual agent optimality interpretation of (1).

[20]We use an infinite time horizon in (2) in order to simplify the exposition. Alternatively, we could use a finite horizon specification together with a bequest function of terminal wealth. The results and implications are exactly the same under the two setups.

increasing and strictly concave in $c(s)$ implying positive but decreasing marginal utility and risk aversion. The investor can trade in various securities in order to carry his wealth through time. It is straightforward to show that the first-order conditions for an optimal consumption and portfolio selection policy imply the following relation for the price $V(t)$ of any asset[21]

$$u_c\left(c^*, K, t\right) V(t) = E_t^P\left[\int_t^{\infty} u_c\left(c^*, K, s\right) A(K, s)\, ds\right]. \tag{3}$$

Here $u_c\left(c^*, K, s\right)$ is the partial derivative of the instantaneous utility function at time s at the optimal aggregate consumption rate $c^*(s)$ with respect to $c(s)$, i.e. marginal utility. Defining $\Lambda(K, s) \equiv u_c\left(c^*, K, s\right)$ highlights the equivalence of (3) and (1). This is again a strong result. The assumption that the investor pursues an optimal consumption and portfolio selection policy directly implies that the prices of all securities satisfy the fundamental asset pricing equation (1) with the stochastic discount factor given by

$$\frac{\Lambda(K, s)}{\Lambda(K, t)} = \frac{u_c\left(c^*, K, s\right)}{u_c\left(c^*, K, t\right)}. \tag{4}$$

The general equilibrium approach to the fundamental asset pricing equation can also be interpreted economically. (3) captures the standard marginal condition for an optimum in finance or economics. It states that today's loss in marginal utility induced by additionally purchasing an infinitesimal amount of the asset, i.e. the left-hand side of (3), must be equal to the increase in expected future marginal utility due to the additional cash flows of the infinitesimal amount in all future states and at all future points in time, i.e. the right-hand side of (3). The stochastic discount factor (4) implied by the general equilibrium approach also has an economic meaning. It is the intertemporal marginal rate of substitution between future aggregate consumption at time s and current aggregate consumption at time t. Thus, the marginal rate of substitution is a valid stochastic discount factor. In contrast to the stochastic discount factor in the no-arbitrage approach, the marginal rate of substitution cannot be observed or inferred from the prices of traded assets. Instead, the general equilibrium approach to asset pricing necessitates the exogenous specification of the representative investor's utility function. Solving the investor's consumption and portfolio selection problem then allows to derive the investor's marginal rate of substitution as the appropriate stochastic discount factor.

3.3 Generalized Continuous-Time SDF Framework

The fundamental asset pricing equation (1) incorporates many asset pricing models as special cases and is valid for a wide variety of securities. However, it has one critical shortcoming: it is only applicable to assets whose cash flows derive from exogenously specified stochastic processes. Securities whose cash flows can be controlled by the investor cannot be described using the standard

[21] See e.g. Ingersoll [1987, pp. 329–330] or Cochrane [2001, pp. 29–30].

asset pricing equation (1). This limitation arises since for these assets the cash flow rate at any time t, $A(K,t)$, depends on the action chosen by the investor at time t and possibly also on the entire history of actions up to time t. Yet, it is unclear which actions the investor takes up to time t and how the equation could capture the owner's control. At first glance, this might seem to be a negligible deficiency as the cash flows of many traded assets cannot be controlled by the owner of the asset. An exception are American-style options. Their cash flows can be influenced by the investor since it is at his discretion when to exercise the option and therefore when to receive the option's payoff. Thus, it is impossible to model American options within the framework of (1).

Given that American-style securities are specialized financial instruments, investor interactions might be of limited interest in standard asset pricing. Yet, they become highly relevant in the context of firm valuation. Many firm valuation approaches from the contingent claims, the real options, and the asset pricing literature explicitly allow for investors' endogenous control of the company's cash flows, e.g. by choosing the firm's investment, financing, or default policies, by temporarily shutting the firm, etc.[22] These models cannot be captured within the standard equation (1). Therefore, we develop a generalization of the fundamental asset pricing equation that can deal with these issues.

We start with some technical details. Assume that the k-dimensional state variable process $K(t,m)$, the discount factor process $\Lambda(K,t,m,n)$, and the cash flow process $A(K,t,m,o)$ are controlled stochastic processes. Here $m \in \mathbb{M}(K,t) \subseteq \mathbb{R}^u$, $n \in \mathbb{N}(K,t) \subseteq \mathbb{R}^v$, $o \in \mathbb{O}(K,t) \subseteq \mathbb{R}^w$ for all $(K,t) \in \mathbb{R}^k \times [0,\infty)$ represent controls determined by control laws $m = M(K,t)$, $n = N(K,t)$, and $o = O(K,t)$.[23] $M(K,t)$ is a u-dimensional, $N(K,t)$ a v-dimensional, and $O(K,t)$ a w-dimensional control process. Assume further that the classes $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{O}$ of admissible control policies are restricted to $\mathcal{F}_t$-adapted processes $M(K,t)$, $N(K,t)$, and $O(K,t)$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{K(s,m) : s \leq t\}$. Finally, assume that the control laws $M(K,t)$, $N(K,t)$, and $O(K,t)$ as well as the controlled processes $K(t,m)$, $\Lambda(K,t,m,n)$, and $A(K,t,m,o)$ satisfy certain regularity conditions.[24]

The economic implications of these assumptions are as follows. By choosing controls m, n, and o at each instant t, it is possible to control the state variable process $K(t,m)$, the stochastic discount factor process $\Lambda(K,t,m,n)$, and the cash flow process $A(K,t,m,o)$, i.e. to influence their current or future values.[25] We distinguish between controls m, n, and o. Here m denotes the control of the state variable process $K(t,m)$, n the control of the discount factor process $\Lambda(K,t,m,n)$ given the controlled state variable process $K(t,m)$, and o the

[22] See e.g. Black and Cox [1976], Brennan and Schwartz [1982b], and Brennan and Schwartz [1985].

[23] Note that we use M, N, and O to denote the control laws while the values of the controls determined by these functions are denoted m, n, and o, respectively.

[24] If $K(t,m)$, $\Lambda(K,t,m,n)$, and $A(K,t,m,o)$ are continuous stochastic processes, the standard growth and Lipschitz conditions as given in Karatzas and Shreve [1991, pp. 287–290] are sufficient conditions. For less restrictive but sufficient regularity conditions, see Fleming and Rishel [1975] and Krylov [1980].

[25] In general, the discount factor process will not be controllable. However, we employ here the most general notation in order to capture any feasible setup.

control of the cash flow process $A(K,t,m,o)$ given the controlled state variable process $K(t,m)$. Thus, while m controls the evolution of the state variables, n controls the mapping of the state variables in the discount factor function and o the mapping of the state variables in the cash flow function. This notation allows for an easy differentiation between complete and incomplete market set-ups. The control n is relevant only in incomplete markets since only there the discount factor is not unique and can be controlled. In a complete market, the discount factor is uniquely specified given the controlled state variables $K(t,m)$, so there exists no control n. The controlled state variable process $K(t,m)$ directly implies that the cash flow process $A(K,t,m,o)$ also follows a controlled stochastic process. The control o additionally allows for controlling the mapping of the controlled state variables in the cash flow function.[26] Thus, the control m influences the discount factor and the cash flow process while the control n affects only the discount factor process and the control o only the cash flow process. The limitation of the classes of admissible control laws implies that only the information revealed up to but not including time t can be used in choosing the controls m, n, and o at time t. The generated σ-algebra $\mathcal{F}_t$ describes the available information up to time t and the measurability of the stochastic processes $M(K,t)$, $N(K,t)$, and $O(K,t)$ with respect to $\mathcal{F}_t$ ensures that only this information is used in determining the controls. Thus, $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{O}$ exclude those control policies $M(K,t)$, $N(K,t)$, and $O(K,t)$ that would be anticipatory, i.e. that would use future information in deriving today's controls. The admissible control policies are called feedback policies. Finally, the regularity conditions ensure that the problem is well-defined and that there exists a solution.

Under these assumptions, the generalized fundamental asset pricing equation in continuous time is given by

$$V(K,t) = \sup_{M,N,O} E_t^P \left[\int_t^\infty \frac{\Lambda(K,s,M,N)}{\Lambda(K,t,M,N)} A(K,s,M,O)\, ds \right], \qquad (5)$$

where $\sup_{M,N,O}$ denotes the supremum operator over the sets of admissible control policies.[27] The assumptions outlined above ensure that there exists a well-behaved unique solution of (5). According to (5), the value of an asset with controlled cash flow process $A(K,t,m,o)$ in an economy with controlled state variables $K(t,m)$ and a controlled discount factor process $\Lambda(K,t,m,n)$ is given by the expected value under the measure P of the integral over the asset's cash flow rate deflated with the stochastic discount factor where the control policies $M(K,t)$, $N(K,t)$, and $O(K,t)$ are chosen optimally, i.e. as to maximize the asset's value. (5) constitutes a substantial generalization of the standard asset pricing framework in the literature. It extends the standard SDF approach by allowing for investor interaction through controlled processes for the state variables, the discount factor, and the cash flow rate. This enables

[26] The controlled cash flow process $A(K,t,m,o)$ can be considered as a generalization of the technology-choice process defined by Duffie [2001, p. 271].

[27] The supremum and not the maximum operator is used in (5) since the maximum of (5) might have measure zero or a finite maximum might not exist.

the analysis of many setups and securities that cannot be treated in the standard framework. Furthermore, it integrates various attempts in the literature that analyze particular setups or particular securities in which controlled state variables, discount factor, or cash flow processes arise.[28]

The generalized fundamental asset pricing equation can be derived along the same two lines as the standard equation (1). First, one can assume an arbitrage-free but incomplete capital market with traded assets whose controllable cash flows are functions of controllable state variables. The arguments of Harrison and Kreps [1979] then directly lead to the above equation. Note that we assumed the capital market to be incomplete. This assumption is critical for that investors might control the discount factor process. If the capital market were complete, the stochastic discount factor would be unique. In this case, investors would not be able to choose the discount factor process but only the state variable and cash flow processes leading to a special case of (5). The arbitrage-free but incomplete market setup is analyzed by Cochrane and Saá-Requejo [2000]. They calculate the lower and upper bounds for the value of an asset by minimizing respectively maximizing over the set of admissible discount factors which they restrict by imposing a volatility bound on the stochastic discount factor. However, Cochrane and Saá-Requejo [2000] only deal with uncontrolled state variables and uncontrolled cash flows. Their equation thus constitutes a special case of the generalized fundamental asset pricing equation (5).[29]

Second, (5) follows from a general equilibrium approach. Consider the consumption and portfolio selection problem of a representative investor who can control his instantaneous utility function $u(c, K, t, m, n)$ as well as the cash flow function of at least one asset. Setting up the investor's portfolio selection problem and determining the first-order conditions yields (5) for any controllable asset. Thus, the generalized fundamental asset pricing equation can be derived from a no-arbitrage perspective as well as from a general equilibrium point of view using the same arguments that lead to the standard equation (1). The only differences concern the assumptions about available assets and the investor's utility function.

A typical application of (5) is the valuation of American-style options in an arbitrage-free and complete capital market. In this case, the state variable and discount factor processes are uniquely specified, but the investor can control the cash flow of the asset. Specifically, the investor's control is given by the stopping time t_S at which he exercises the option. This notation, which constitutes a special case of (5), is e.g. used by Duffie [2001].

3.4 Generalized SDF Framework and Firm Valuation

So far, we have extended the fundamental asset pricing equation to cope with controlled state variable, discount factor, and cash flow processes in asset pricing. However, we are mainly interested in the valuation of companies. Therefore, we now argue that (5) should be the basis of any firm valuation model.

[28]See e.g. Cochrane and Saá-Requejo [2000] and Duffie [2001].

[29]See Cochrane and Saá-Requejo [2000, equation (26)].

According to Cochrane [2001], "asset pricing theory tries to understand the prices or values of claims to uncertain payments."[30] This definition could also describe the objective of the firm valuation literature. Namely, this literature is concerned with the prices or values of corporate securities whose future payments are obviously uncertain. For example, the payments on a firm's debt are uncertain as the firm might default on its obligations. The payments to equityholders, i.e. in particular dividends, are not even contractually specified and thus uncertain as well. It hence seems natural to consider firm valuation as a particular area of asset pricing that deals with corporate securities.

Keeping this in mind, we can reinterpret the generalized fundamental asset pricing equation. For the time being, neglect the supremum operator in (5). Further, assume that the cash flow rate $A(K, t, m, o)$ describes the dividends paid by an entirely equity-financed firm. (5) can then be considered as a *Dividend Discount Model* of firm valuation. Alternatively, let $A(K, t, m, o)$ be the free cash flows of the entirely equity-financed firm and (5) constitutes a *Discounted Cash Flow Model* of firm valuation. It is obvious that these interpretations can be extended to any other corporate security if the firm has a more complicated capital structure. (5) can thus also be regarded as the fundamental firm valuation equation. The flexibility to integrate different firm valuation approaches is one advantage of the SDF framework. Yet, the framework also offers various other advantages.

First, it highlights the correspondence between firm valuation and asset pricing. While firm valuation is often considered part of corporate finance, equation (5) stresses that it could as well be classified as part of asset pricing.

Second, the generalization of the fundamental asset pricing equation constitutes a natural framework for the formulation of contingent claims, real options, and asset pricing approaches to firm valuation. Many models from these strands of literature include endogenous decisions of security holders or the firm's management affecting the firm's state variables and cash flows. The generalized fundamental asset pricing equation provides the first coherent framework to model such decisions through the introduction of controlled processes.

Third, equation (5) allows to integrate no-arbitrage and general equilibrium models of firm valuation within the same framework. The difference between these classes only arises from the discount factor process. In no-arbitrage approaches, the discount factor process is specified exogenously or derived from existing assets. Equilibrium models in contrast deduce the discount factor from investors' preferences. Furthermore, the generalized fundamental asset pricing equation facilitates the exposition of general equilibrium models of firm valuation. Within the SDF framework one can separate the derivation of the stochastic discount factor process from the specification of the firm's cash flow process. In existing frameworks, these steps are not that clearly distinguishable.[31]

Fourth, the concepts of controlled state variable and cash flow processes and the resulting valuation equation (5) allow for an easy formalization of issues that

[30] Cochrane [2001, p. xiii].

[31] See e.g. Brennan and Schwartz [1982a,b, 1984].

can hardly be captured in other approaches to firm valuation. As an example, consider corporate restructurings such as mergers and acquisitions. In such transactions, the acquirer generally pays a price for the target company well above the target's market value before the transaction. This phenomenon is often explained by realizable synergies or the increased control of the acquirer for which a control premium must be paid. Within the framework of (5) the loose notions of increased control or synergies can be given a mathematically precise meaning. Namely, they constitute extensions of the sets of admissible control policies $\mathcal{M}$ and $\mathcal{O}$ that affect the firm's cash flows. In case the extended sets of control policies contain at least one control law that leads to higher cash flows, the control premium might be justified. Standard frameworks to firm valuation are hardly able to deal with such concepts.

Fifth, (5) is suited to also deal with the subjective, marginal utility-based approach to firm valuation which is particularly relevant in the German-speaking literature. This literature argues that there does not exist an objective market value of a firm but only subjective values specific to each individual and the objective of the valuation. The literature further stresses the differences between this approach and the no-arbitrage or general equilibrium models of firm valuation. However, the individualistic approach to firm valuation is also contained in the generalized fundamental asset pricing equation (5). Define the stochastic discount factor as an individual agent's intertemporal marginal rate of substitution and let the classes of admissible control laws be given by the choices available to the individual agent. Under these assumptions, (5) yields the individual agent's specific firm value.

Given the generality and the various advantages of the generalized fundamental asset pricing equation, we take it as the basis of all firm valuation models in continuous time. Thereby, the general equation (5) delivers an important insight. Namely, it highlights that all firm valuation models can at most differ with respect to 6 dimensions.

i. The state variables can differ from model to model.

ii. The derivation of the discount factor $\Lambda(K, s, m, n)/\Lambda(K, t, m, n)$ can vary between the models. We differentiate between no-arbitrage and general equilibrium models, i.e. between methodologies with exogenously determined and those with endogenously determined discount factors.

iii. The specification, i.e. the factor structure, of the discount factor process can be chosen differently.

iv. The specification of the cash flow function $A(K, t, m, o)$ depends on the corporate security to be valued. In the case of equity, there exist two possible specifications for $A(K, t, m, o)$. We can consider $A(K, t, m, o)$ as the firm's dividend payments making (5) a *Dividend Discount Model*. In this case, we have to make explicit assumptions about the firm's dividend policy. Alternatively, $A(K, t, m, o)$ can designate the firm's free cash flows less the payouts to other securities. Then (5) constitutes a *Discounted Cash Flow Model* and we do not have to make any assumptions about the dividend policy.

v. The sets of admissible control laws $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{O}$ depend on the state variables, the assumed decision rights of the security holders and the firm's management, and the completeness of the capital market.

vi. Finally, the boundary conditions are determined by the corporate security to be valued and the model's assumptions.

The existing continuous-time firm valuation models in the literature can all be derived as special cases of the generalized fundamental asset pricing equation by imposing their respective assumptions on (5). Thereby, the specific assumptions of the various models address exactly the 6 dimensions outlined above.

3.5 Fundamental Differential Equations

The standard fundamental asset pricing equation (1) can be solved in two ways.[32] First, given assumptions about the state variable process $K(t)$, the discount factor process $\Lambda(K,t)$, and the cash flow process $A(K,t)$, one can solve these processes forward for the discount factor and the cash flows in each state of the world and at each instant of time. Then one evaluates the integral and the expectation operator to find the value of the corporate security $V(K,t)$. Second, one can derive a PDE from (1) that the value of any non-controllable asset $V(K,t)$ must satisfy. Imposing assumptions about the discount factor process and the cash flow process on the PDE yields a functional equation for $V(K,t)$. Together with the boundary conditions, one can solve the PDE backwards for the value of the corporate security $V(K,t)$.

While the fundamental asset pricing equation can be solved in either of the two ways, only the second approach is applicable to the generalized fundamental asset pricing equation. This is due to the fact that (5) constitutes a stochastic control problem which has to be solved using stochastic control theory. In general, it would be feasible to apply the specific assumptions of any firm valuation model on (5) and then to derive the differential equation specific to this model. Yet, it seems more sensible to derive a general differential equation from (5) and to impose the specific assumptions of any firm valuation model directly on the differential equation. The general differential equation will thus be our starting point to show how any particular firm valuation model can be developed as a special case of (5).

We first derive the differential equation without imposing specific assumptions on the state variable process $K(t,m)$, the discount factor process $\Lambda(K,t,m,n)$, and the value function $V(K,t)$. This enables us to compare our findings to some results in the asset pricing literature. Then we restrict the admissible processes for $K(t,m)$ and $\Lambda(K,t,m,n)$ and the admissible class of value functions $V(K,t)$. These assumptions allow the derivation of a more specific PDE. The resulting PDE is at the core of all existing firm valuation models in continuous time.

[32] See Cochrane [2001].

The differential equation for any asset value $V(K,t)$ that can be derived from (5) without imposing further conditions on $K(t,m)$, $\Lambda(K,t,m,n)$, and $V(K,t)$ is given in the following proposition.

Proposition 1 *Suppose that the assumptions of section 3.3 hold. Define*

$$V(K,t) = \sup_{M,N,O} Q(K,t,M,N,O), \tag{6}$$

where

$$Q(K,t,M,N,O) = E_t^P\left[\int_t^\infty \frac{\Lambda(K,s,M,N)}{\Lambda(K,t,M,N)} A(K,s,M,O)\, ds\right]$$

and $M(K,t) \in \mathcal{M}$, $N(K,t) \in \mathcal{N}$, $O(K,t) \in \mathcal{O}$ are admissible control laws. Suppose that there exist optimal admissible control laws $M^*(K,t)$, $N^*(K,t)$, and $O^*(K,t)$ such that $V(K,t) = Q(K,t,M^*,N^*,O^*) = \sup_{M,N,O} Q(K,t,M,N,O)$. The value function $V(K,t)$ then satisfies

$$\sup_{m,n,o} E_t^P\left[A(K,t,m,o)\, dt + \frac{1}{\Lambda(K,t,m,n)}\, d(\Lambda(K,t,m,n)V(K,t))\right] = 0 \qquad \forall (K,t) \in \mathbb{R}^k \times [0,\infty). \tag{7}$$

For each point $(K,t) \in \mathbb{R}^k \times [0,\infty)$, the supremum in (7) is attained for $m = M^*(K,t)$, $n = N^*(K,t)$, and $o = O^*(K,t)$.

Proof. See appendix A.1. □

In an economy with controlled state variable, discount factor, and cash flow processes, (7) constitutes the fundamental differential equation to asset pricing. It closely resembles the fundamental differential equation for uncontrolled processes as e.g. derived by Cochrane [2001, equation (1.29)]. Economically, it implies that iff the state variable, the discount factor, and the cash flow processes are managed optimally, deflated asset prices are martingales under P with the stochastic discount factor as numéraire. The first term in (7) thereby adjusts for the instantaneous cash flow of the asset. As becomes clear from the proof in appendix A.1, in case the control laws are chosen sub-optimally, deflated asset prices are supermartingales under P, i.e. they have a negative trend. In a general equilibrium model, the asset would then yield a below equilibrium rate of return. Since all existing firm valuation models are based on assumptions about the state variable process $K(t,m)$, the discount factor process $\Lambda(K,t,m,n)$, and the cash flow process $A(K,t,m,o)$, we transform (7) into an equivalent equation on which specific process assumptions can be applied more conveniently.

Proposition 2 *Suppose that the assumptions of proposition 1 hold and that there exists a risk-less asset paying the risk-free rate of interest $r(K,t,m)$. The* *value function* $V(K,t)$ *then satisfies*

$$\sup_{m,n,o}\Bigg[A(K,t,m,o)\,dt + E_t^P[dV(K,t)] - r(K,t,m)V(K,t)\,dt + E_t^P\left[\frac{d\Lambda(K,t,m,n)}{\Lambda(K,t,m,n)}\,dV(K,t)\right]\Bigg] = 0 \qquad (8)$$
$$\forall (K,t) \in \mathbb{R}^k \times [0,\infty).$$

For each point $(K,t) \in \mathbb{R}^k \times [0,\infty)$*, the supremum in* (8) *is attained for* $m = M^*(K,t)$*,* $n = N^*(K,t)$*, and* $o = O^*(K,t)$*.*

Proof. See appendix A.2. □

Equation (8) constitutes an alternative differential equation to asset pricing in an economy with controlled processes under the additional assumption that there exists a risk-less asset. Its counterpart in a world with uncontrolled processes can be found e.g. in Cochrane [2001, equation (1.35)]. The economic intuition of (8) is most clearly visible if one divides (8) by $V(K,t)$. In this case, (8) becomes an expected return relation. It then states that the instantaneous expected rate of return of any optimally managed corporate security, i.e. the first two terms on the left-hand side of (8), equals the instantaneous risk-free rate of interest $r(K,t,m)$ plus a risk adjustment which is given by the covariance of the asset's return process with the discount factor process, i.e. the last term on the left-hand side. Thus, (8) extends a standard asset pricing result to an economy with controlled state variable, discount factor, and cash flow processes.

So far, we have not made specific assumptions about the state variable process, the discount factor process, and the value function. Consequently, the derived differential equations (7) and (8) are very general from a mathematical point of view. In addition, it is not clear if and how (7) and (8) can be solved. We therefore now restrict the admissible processes $K(t,m)$ to the class of Itô-Poisson processes and the admissible processes $\Lambda(K,t,m,n)$ to the class of Itô processes as these are the predominant process specifications in financial economics. Furthermore, we assume that the value function $V(K,t) \in \mathcal{C}^{2,1}(\mathbb{R}^k \times [0,\infty))$. Here $\mathcal{C}^{2,1}(\mathbb{R}^k \times [0,\infty))$ denotes the class of functions that are twice continuously differentiable with respect to the first argument and once with respect to the second argument. These assumptions allow us to derive a more specific fundamental PDE that is satisfied by any existing asset pricing and firm valuation model. Moreover, it becomes clear how the PDE can be solved using a well-known mathematical tool, namely stochastic control theory. The fundamental PDE under the afore-mentioned assumptions is given in the following proposition.

Proposition 3 *Suppose that the assumptions of proposition 1 hold and that there exists a risk-less asset paying the risk-free rate of interest* $r(K,t,m)$*. Define*

$$dK(t,m) = \mu_K(K,t,m)\,dt + \Sigma_K(K,t,m)\,dz_K(t) + \Upsilon_K(K,t,m)\,dq_K(t), \qquad (9)$$

where $K \in \mathbb{R}^k$, $\mu_K : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^u \to \mathbb{R}^k$, $\Sigma_K : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^u \to \mathbb{R}^{k \times x_K}$, $\Upsilon_K : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^u \to \mathbb{R}^{k \times y}$, $z_K(t)$ *is a* x_K*-dimensional standard Wiener process, and* $q_K(t)$ *is a* y*-dimensional Poisson process with intensity vector* $\lambda_K(K,t,m)$. *Further, define*

$$\frac{d\Lambda(K,t,m,n)}{\Lambda(K,t,m,n)} = -r(K,t,m)\,dt - \sigma_\Lambda^T(K,t,m,n)\,dz_\Lambda(t), \tag{10}$$

where $\Lambda \in \mathbb{R}$, $r : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^u \to \mathbb{R}$, $\sigma_\Lambda : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^u \times \mathbb{R}^v \to \mathbb{R}^{x_\Lambda \times 1}$, *and* $z_\Lambda(t)$ *is a* x_Λ*-dimensional standard Wiener process. Suppose that the value function* $V(K,t) \in \mathcal{C}^{2,1}(\mathbb{R}^k \times [0,\infty))$. *The value function* $V(K,t)$ *then satisfies*

$$\begin{aligned}\sup_{m,n,o} \Big[&A(K,t,m,o) + L^{m,n,o}[V(K,t)] - r(K,t,m)V(K,t) \\ &- \sigma_\Lambda^T(K,t,m,n)\rho_{K\Lambda}^T \Sigma_K^T(K,t,m) V_K(K,t) \Big] + V_t(K,t) = 0 \\ &\qquad\qquad\qquad\qquad \forall (K,t) \in \mathbb{R}^k \times [0,\infty).\end{aligned} \tag{11}$$

For each point $(K,t) \in \mathbb{R}^k \times [0,\infty)$, *the supremum in* (11) *is attained for* $m = M^*(K,t)$, $n = N^*(K,t)$, *and* $o = O^*(K,t)$.

Proof. See appendix A.3. □

In (11) and throughout the paper, subscripts on $V(K,t)$ represent partial derivatives, $\rho_{K\Lambda}$ the $x_K \times x_\Lambda$-dimensional correlation matrix between the standard Wiener processes $z_K(t)$ and $z_\Lambda(t)$, and $L^{m,n,o}[F(K,t)]$ the partial differential operator for Itô-Poisson processes

$$\begin{aligned}L^{m,n,o}[F(K,t)] =& \mu_K^T(K,t,m)F_K(K,t) \\ &+ \lambda_K^T(K,t,m)\,E_t^P\left[(F(K+v_{\cdot i}\theta_i,t) - F(K,t))_{i=1}^y\right] \\ &+ \frac{1}{2}\,\mathrm{tr}\left[\Sigma_K(K,t,m)\Sigma_K^T(K,t,m)F_{KK}(K,t)\right],\end{aligned} \tag{12}$$

where $(F(K+v_{\cdot i}\theta_i,t) - F(K,t))_{i=1}^y$ denotes a y-dimensional vector, $v_{\cdot i}(K,t,m)$ the ith column of $\Upsilon_K(K,t,m)$, $\theta_i(t)$ the jump size of the ith Poisson process, and $\mathrm{tr}[\cdot]$ the trace of a matrix. j^T means the transpose of a vector or a matrix j. The expectation operator $E_t^P[\cdot]$ arises in (12) as the jump heights $\theta_i(t)$ can be random variables.

Equation (11) is satisfied by any existing asset pricing and firm valuation model in continuous time whose state variables follow Itô-Poisson processes and whose discount factor follows an Itô process. It thus comprises e.g. the fundamental PDEs derived by Merton [1977, equation (1)] and Cox, Ingersoll, and Ross [1985b, equation (31)] as special cases. Besides its fundamental role in asset pricing and firm valuation, (11) is also interesting from a mathematical point of view. The PDE (11) constitutes a standard Hamilton-Jacobi-Bellman (HJB) equation common in stochastic control theory. Thus, by restricting the classes of admissible state variable and discount factor processes, we are able to transform the asset pricing equations (5), (7), and (8) into a well-defined stochastic control problem.

Yet, propositions 1–3 are only necessary conditions. Proposition 3 e.g. only states that if a value function $V(K,t)$ and optimal control policies $M^*(K,t)$, $N^*(K,t)$, $O^*(K,t)$ exist, then $V(K,t)$ satisfies (11) and $M^*(K,t)$, $N^*(K,t)$, $O^*(K,t)$ realize the supremum in this equation. It remains unclear whether the respective conditions of propositions 1–3 are also sufficient, i.e. whether a function $W(K,t)$ and control policies $R(K,t)$, $S(K,t)$, $T(K,t)$ that satisfy the conditions constitute the value function and the optimal control policies. Due to the generality of propositions 1 and 2, it is impossible to prove the sufficiency of the conditions in these cases. However, it is a well-known result that the HJB equation of proposition 3 also represents a sufficient condition for the optimal control problem. This so-called verification theorem implies that if there exists a function $W(K,t)$ that satisfies (11) and for each $(K,t) \in \mathbb{R}^k \times [0,\infty)$ the supremum in (11) is attained for admissible control laws $R(K,t)$, $S(K,t)$, $T(K,t)$, then $W(K,t)$ is indeed the value function $V(K,t) = W(K,t) = Q(K,t,R,S,T)$ and $R(K,t)$, $S(K,t)$, $T(K,t)$ are the optimal control laws $M^*(K,t) = R(K,t)$, $N^*(K,t) = S(K,t)$, and $O^*(K,t) = T(K,t)$. We renounce here on an explicit proof of the verification theorem since it does not provide any new economic insight and the relevant equations are already given in proposition 3.[33]

Proposition 3 and the associated verification theorem reduce the dynamic stochastic control problem to the static maximization of the HJB equation (11). Thereby, they also provide a direct solution approach to the stochastic control problem. Namely, one fixes an arbitrary point $(K,t) \in \mathbb{R}^k \times [0,\infty)$ and derives the first-order conditions of the static optimization problem given by (11) with respect to the controls m, n, and o. Solving the first-order conditions yields optimal controls $m^* = M^*(K,t,V)$, $n^* = N^*(K,t,V)$, and $o^* = O^*(K,t)$ as functions of K, t, and the partial derivatives of the value function $V(K,t)$. Substituting the optimal control laws back into (11) allows us to drop the supremum operator and leads either to a single PDE or to a set of PDEs for $V(K,t)$. The PDE or the set of PDEs can then be solved either analytically or numerically for $V(K,t)$. Having obtained the value function $V(K,t)$, we replace $V(K,t)$ and its partial derivatives in the optimal control policies $M^*(K,t,V)$ and $N^*(K,t,V)$ leaving the control laws only as functions of the state variables $K(t)$ and time t. Due to the verification theorem of proposition 3, $V(K,t)$ must be the value function and $M^*(K,t)$, $N^*(K,t)$, $O^*(K,t)$ must be the optimal control laws.

In general, we could now proceed to derive the existing firm valuation models as special cases of proposition 3. However, compared to the models developed in the financial literature so far, proposition 3 is still very general. It permits that each state variable might be driven by several Wiener and Poisson processes. Yet, the existing firm valuation models all assume that the dynamics of each state variable are determined by at most one Wiener and one Poisson process. Using this more restrictive assumption, we obtain the following result as a specialization of proposition 3.

[33] Proofs of the verification theorem for pure diffusion processes can be found in Fleming and Rishel [1975], Gihman and Skorohod [1979], and Krylov [1980]. For jump-diffusion processes, see Merton [1971] and the references therein.

Proposition 4 *Suppose that the assumptions of proposition 3 hold. Further, suppose that* $\Sigma_K(K,t,m) \in \mathbb{R}^{k\times k}$ *and* $\Upsilon_K(K,t,m) \in \mathbb{R}^{k\times k}$ *are diagonal matrices, i.e.*

$$dK_i(t,m) = \mu_{K_i}(K,t,m)\,dt + \sigma_{K_i}(K,t,m)\,dz_{K_i}(t) + \upsilon_{K_i}(K,t,m)\,dq_{K_i}(t) \qquad \forall\, i \in \{1,\dots,k\}, \quad (13)$$

where $K_i \in \mathbb{R}$, $\mu_{K_i} : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^u \to \mathbb{R}$, $\sigma_{K_i} : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^u \to \mathbb{R}$, $\upsilon_{K_i} : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^u \to \mathbb{R}$, $z_{K_i}(t)$ *is a one-dimensional standard Wiener process with* $dz_{K_i}(t)\,dz_{K_j}(t) = \rho_{K_iK_j}\,dt$, *and* $q_{K_i}(t)$ *is a one-dimensional Poisson process with intensity* $\lambda_{K_i}(K,t,m)$. *The value function* $V(K,t)$ *then satisfies*

$$\sup_{m,n,o}\Bigg[A(K,t,m,o) + L^{m,n,o}[V(K,t)] - r(K,t,m)V(K,t) - \sum_{i=1}^{k}\sum_{j=1}^{x_\Lambda}\sigma_{K_i}(K,t,m)\sigma_{\Lambda_j}(K,t,m,n)\rho_{K\Lambda_{ij}}V_{K_i}(K,t)\Bigg] + V_t(K,t) = 0 \qquad \forall (K,t) \in \mathbb{R}^k \times [0,\infty). \quad (14)$$

For each point $(K,t) \in \mathbb{R}^k \times [0,\infty)$, *the supremum in* (14) *is attained for* $m = M^*(K,t)$, $n = N^*(K,t)$, *and* $o = O^*(K,t)$.

Proof. See appendix A.4. □

In (14), $\rho_{K\Lambda_{ij}}$ denotes the correlation coefficient between the standard Wiener processes $z_{K_i}(t)$ and $z_{\Lambda_j}(t)$ and $L^{m,n,o}[F(K,t)]$ the partial differential operator for Itô-Poisson processes

$$L^{m,n,o}[F(K,t)] = \sum_{i=1}^{k}\mu_{K_i}(K,t,m)\,F_{K_i}(K,t) + \sum_{i=1}^{k}\lambda_{K_i}(K,t,m)\,E_t^P\left[F(K + \iota_i\upsilon_{K_i}\theta_{K_i},t) - F(K,t)\right] + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\sigma_{K_i}(K,t,m)\sigma_{K_j}(K,t,m)\rho_{K_iK_j}\,F_{K_iK_j}(K,t), \quad (15)$$

where ι_i denotes a k-dimensional vector with a 1 in the ith row and zeroes in all other positions. Since proposition 4 constitutes a special case of proposition 3, there also exists a verification theorem for proposition 4.

Despite the various assumptions we imposed in order to derive proposition 4, it is still general enough to capture most asset pricing and all firm valuation models known in the literature. We therefore take (14) as our starting point in order to derive the existing firm valuation models as special cases of the generalized asset pricing equation (5). Yet, it should be kept in mind that (5) and its differential form (7) actually are the most fundamental asset pricing equations in continuous time. They are at the core of any continuous-time firm valuation model. Equation (8) additionally requires the existence of a risk-less asset and (11) and (14) obtain iff one restricts the classes of admissible state variable and discount factor processes. We are only able to derive all existing firm valuation models as special cases of (14) as the existing models all share the more restrictive assumptions of proposition 4.

4 No-Arbitrage Models

In this section, we derive the existing no-arbitrage models of firm valuation as special cases of the generalized fundamental asset pricing equation. The no-arbitrage models of firm valuation share the assumption of an arbitrage-free and complete capital market implying the existence of a unique stochastic discount factor. Building on this relation, the no-arbitrage models specify the discount factor exogenously. The existing no-arbitrage models of firm valuation are Gordon [1962] and Bakshi and Chen [2001]. We first discuss the basic Gordon Growth Model and then analyze the three-factor model of Bakshi and Chen [2001].

4.1 Gordon [1962]

The Gordon Growth Model can be considered as the first parametric model of firm valuation. Its only state variable are the firm's earnings $Y(t)$ which are assumed to grow at a constant rate ν, i.e. they follow the deterministic process

$$dY(t) = \nu Y(t)\, dt. \tag{16}$$

Gordon [1962] specifies the growth rate in earnings ν to be $\nu = (1 - \alpha)\beta$. Thereby, α denotes the payout rate, i.e. the percentage of corporate earnings that is paid out as dividends. Thus, $(1 - \alpha)$ stands for the retention rate of the firm's earnings and β for the return on investment that the company can earn on the retained earnings. Both parameters, α and β, are assumed to be constant over time. The earnings process as the single state variable of the model is not controllable. Consequently, the set of admissible controls for the state variables $\mathbb{M}$ is empty, i.e. $\mathbb{M} = \emptyset$.

The cash flow function $A(Y)$ is interpreted as the firm's dividend payments $DI(Y)$. Consequently, the Gordon Growth Model constitutes a *Dividend Discount Model*. Thereby, $A(Y)$ is assumed to have the simple functional form

$$A(Y) = DI(Y) = \alpha Y(t). \tag{17}$$

This entails that a constant fraction α of the firm's earnings $Y(t)$ is paid out as dividends. From (17) it is obvious that investors cannot control the cash flow function so that the set of admissible controls for the cash flow function is empty, i.e. $\mathbb{O} = \emptyset$.

The capital market underlying the Gordon Growth Model is assumed to be arbitrage-free and complete implying the existence of a unique stochastic discount factor. Since the discount factor is uniquely determined, investors cannot control the discount factor process. Thus, the set of admissible controls for the discount factor $\mathbb{N}$ is empty as well, i.e. $\mathbb{N} = \emptyset$. In his original work, Gordon [1962] does not explicitly specify the factor structure of the stochastic discount factor, but he assumes that there exists a risk-less asset paying a constant risk-free rate of interest. Therefore, the discount factor is given by the general discount factor process

$$\frac{d\Lambda(K,t)}{\Lambda(K,t)} = -r\,dt - \sigma_\Lambda^T(K,t)\,dz_\Lambda(t). \tag{18}$$

Finally, it is assumed that the firm's capital structure contains no debt, i.e. the company is entirely equity-financed, and that the firm continues operating forever. Together with (16)–(18), these assumptions completely specify the Gordon Growth Model.

Proposition 5 (Gordon [1962]) *Suppose that the state variable process is given by* (16)*, the cash flow function by* (17)*, and the discount factor process by* (18)*. Suppose further that the firm is entirely equity-financed and continues operating forever. The firm value $V(Y)$ then satisfies the following ordinary differential equation*

$$(1-\alpha)\beta Y(t)V_Y(Y) + \alpha Y(t) - rV(Y) = 0. \tag{19}$$

The solution of (19) *is given by*

$$V(Y) = \frac{\alpha Y(t)}{r-(1-\alpha)\beta} = \frac{DI(Y)}{r-\nu}, \tag{20}$$

subject to the transversality condition $r > \nu$.

Proof. See appendix A.5. □

Equation (20) constitutes the Gordon Growth Model of firm valuation. The first formula on the right-hand side of (20) is equivalent to the notation in Gordon [1962, equation (4.6)] while the second one corresponds to the notation in Gordon and Shapiro [1956, equation (7)]. The transversality condition ensures that the model is well-defined and that the firm value is finite.[34] If the dividend growth rate, which in the setup of Gordon [1962] corresponds to the expected earnings growth rate ν, were greater than the risk-free rate of interest r, the firm value would be infinite and (20) would be ill-defined. Here, we have derived the Gordon Growth Model as a special case of the generalized fundamental asset pricing equation. The model's economic intuition is very simple. It only states that in case a firm's dividend stream grows forever at a constant rate, the firm value is given by a growing perpetuity.

It is noteworthy that the discount rate in the denominator of (20) is the risk-free rate of interest r. The discount rate must be the risk-free rate of interest

[34]For a detailed explanation of the transversality condition, see Ingersoll [1987, pp. 231–234].

as the earnings process was assumed to be deterministic. This implies that all future dividend payments are certain so that they must be discounted at the constant risk-free rate of interest. Although Gordon [1962] observes that under a deterministic earnings process the dividends in any future period are certain, he does not reach the conclusion that the appropriate discount rate must be the risk-free rate of interest.[35]

The Gordon Growth Model provides a simple and intuitive solution, yet it has many shortcomings. First, the earnings process of a company is stochastic and not deterministic. Only in monopolistic or highly regulated industries might a deterministic earnings process be a reasonable approximation of reality. Second, the proposed specification of the earnings process can only deal with positive earnings. Building on a large sample of companies, Chan, Karceski, and Lakonishok [2002] however find that in every year on average 29.0% of all firms display negative earnings. Consequently, the Gordon Growth Model is not applicable for a substantial number of firms. Third, the assumption of a constant risk-free rate of interest is obviously at odds with empirical observations. Fourth, although many companies have target payout ratios, assuming a constant payout rate seems to be too restrictive. Fifth, there is an increasing number of firms that do not pay dividends. As the Gordon Growth Model is based on dividend payments, it cannot be used to value these companies.

4.2 Bakshi and Chen [2001]

While the Gordon Growth Model contains only one state variable, the model of Bakshi and Chen [2001] is a three-factor model. The first state variable is the short rate, i.e. the instantaneous risk-free rate of interest, which is assumed to follow an Ornstein-Uhlenbeck process

$$dr(t) = \kappa_r(\bar{r} - r(t))\,dt + \sigma_r\,dz_r(t). \tag{21}$$

Here $\bar{r}$ denotes the long-term mean of the short rate, κ_r the mean reversion speed parameter, σ_r the time-invariant volatility of the interest rate process, and $z_r(t)$ a standard Wiener process. (21) implies that the short rate varies stochastically around its long-term mean $\bar{r}$. This specification for the interest rate process is consistent with the one-factor term structure model of Vasicek [1977]. In line with Gordon [1962], the second state variable in the model of Bakshi and Chen [2001] is given by the earnings process of the firm

$$dY(t) = \nu(t)Y(t)\,dt + \sigma_Y Y(t)\,dz_Y(t), \tag{22}$$

where the drift $\nu(t)$ denotes the expected growth rate in earnings, σ_Y the constant volatility of the earnings growth rate, and $z_Y(t)$ a standard Wiener process. The third state variable is the expected earnings growth rate $\nu(t)$ itself which is modeled as an Ornstein-Uhlenbeck process

$$d\nu(t) = \kappa_\nu(\bar{\nu} - \nu(t))\,dt + \sigma_\nu\,dz_\nu(t), \tag{23}$$

[35] See Gordon [1962, pp. 44–46].

whereby $\bar{\nu}$ designates the long-term expected earnings growth rate, κ_ν the mean reversion speed parameter, σ_ν the volatility of changes in the expected earnings growth rate, and $z_\nu(t)$ again a standard Wiener process. The specification of the earnings process of Bakshi and Chen [2001] constitutes a substantial generalization of the assumptions of Gordon [1962]. In contrast to Gordon [1962], the earnings process is no longer deterministic but evolves stochastically around its trend. Furthermore, the expected earnings growth rate also follows a stochastic process. This assumption allows the model to capture temporary effects, e.g. in case a firm currently enjoys above-average earnings growth rates due to patent protected products that can be expected to revert back to more sustainable levels. Comparable to Gordon [1962], Bakshi and Chen [2001] also model the state variables as uncontrolled processes. Hence, the set of admissible controls for the state variables $\mathbb{M}$ is empty. Finally, the Wiener process increments $dz_r(t)$, $dz_Y(t)$, and $dz_\nu(t)$ driving the state variables might be correlated. The respective correlation coefficients are given by ρ_{rY}, $\rho_{r\nu}$, and $\rho_{Y\nu}$.

Consistent with Gordon [1962], Bakshi and Chen [2001] define the cash flow function $A(Y, \nu, t)$ as the firm's dividend payments. Thus, the approach of Bakshi and Chen [2001] also belongs to the class of *Dividend Discount Models*. The functional form of $A(Y, t)$ is given by

$$A(Y,t)\,dt = DI(Y,t)\,dt = \alpha Y(t)\,dt + dW(t), \tag{24}$$

with $dW(t)$ denoting the increment of a martingale process $W(t)$. This setup allows for stochastic dividend payments and deviations from the mean payout rate α thereby generalizing the Gordon Growth Model. Bakshi and Chen [2001] further assume that the increment $dW(t)$ is uncorrelated with all Wiener processes. Consequently, $dW(t)$ is completely unsystematic and is not a priced risk factor. Since (24) does not allow for investor interactions, the set of admissible controls for the cash flow function $\mathbb{O}$ is empty.

Finally, Bakshi and Chen [2001] assume an arbitrage-free and complete capital market. This implies the existence of a unique stochastic discount factor. Hence, the set of admissible controls for the discount factor $\mathbb{N}$ is empty as well. Specifically, Bakshi and Chen [2001] propose the following discount factor

$$\frac{d\Lambda(r,t)}{\Lambda(r,t)} = -r(t)\,dt - \sigma_\Lambda\,dz_\Lambda(t), \tag{25}$$

where σ_Λ is the constant volatility of the discount factor and $z_\Lambda(t)$ a standard Wiener process. Bakshi and Chen [2001] do not provide an economic interpretation of the risk factor $dz_\Lambda(t)$. The Wiener process increments $dz_r(t)$, $dz_Y(t)$, $dz_\nu(t)$, and the risk factor increment $dz_\Lambda(t)$ might be correlated with the correlation coefficients given by $\rho_{r\Lambda}$, $\rho_{Y\Lambda}$, and $\rho_{\nu\Lambda}$, respectively. The assumptions laid out in (21)–(25) lead to the following result.[36]

[36]In (26) and throughout the paper, we drop the functional arguments if this simplifies the notation without inducing any ambiguities.

Proposition 6 (Bakshi and Chen [2001]) *Suppose that the state variable processes are given by* (21)–(23)*, the cash flow function by* (24)*, and the discount factor process by* (25)*. Suppose further that the firm is entirely equity-financed and continues operating forever. The firm value $V(r, Y, \nu)$ then satisfies the following partial differential equation*

$$\begin{aligned}
&\frac{1}{2}\sigma_r^2 V_{rr} + \frac{1}{2}\sigma_Y^2 Y^2(t) V_{YY} + \frac{1}{2}\sigma_\nu^2 V_{\nu\nu} + \sigma_r\sigma_Y\rho_{rY} Y(t) V_{rY} \\
&+ \sigma_r\sigma_\nu\rho_{r\nu} V_{r\nu} + \sigma_Y\sigma_\nu\rho_{Y\nu} Y(t) V_{Y\nu} \\
&+ \left(\kappa_r(\bar{r} - r(t)) - \sigma_r\sigma_\Lambda\rho_{r\Lambda}\right) V_r + \left(\nu(t) - \sigma_Y\sigma_\Lambda\rho_{Y\Lambda}\right) Y(t) V_Y \\
&+ \left(\kappa_\nu(\bar{\nu} - \nu(t)) - \sigma_\nu\sigma_\Lambda\rho_{\nu\Lambda}\right) V_\nu + \alpha Y(t) - r(t) V = 0.
\end{aligned} \tag{26}$$

The solution of (26) *is given by*

$$V(r, Y, \nu) = \alpha Y(t) \int_0^\infty e^{a(s) - b(s) r(t) + c(s)\nu(t)}\, ds, \tag{27}$$

where

$$\begin{aligned}
a(s) =& -\sigma_Y\sigma_\Lambda\rho_{Y\Lambda} s + \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\left(s + \frac{1 - e^{-2\kappa_r s}}{2\kappa_r} - \frac{2\left(1 - e^{-\kappa_r s}\right)}{\kappa_r}\right) \\
&- \frac{\kappa_r\bar{r} - \sigma_r\sigma_\Lambda\rho_{r\Lambda} + \sigma_r\sigma_Y\rho_{rY}}{\kappa_r}\left(s - \frac{1 - e^{-\kappa_r s}}{\kappa_r}\right) \\
&+ \frac{1}{2}\frac{\sigma_\nu^2}{\kappa_\nu^2}\left(s + \frac{1 - e^{-2\kappa_\nu s}}{2\kappa_\nu} - \frac{2\left(1 - e^{-\kappa_\nu s}\right)}{\kappa_\nu}\right) \\
&+ \frac{\kappa_\nu\bar{\nu} - \sigma_\nu\sigma_\Lambda\rho_{\nu\Lambda} + \sigma_Y\sigma_\nu\rho_{Y\nu}}{\kappa_\nu}\left(s - \frac{1 - e^{-\kappa_\nu s}}{\kappa_\nu}\right) \\
&- \frac{\sigma_r\sigma_\nu\rho_{r\nu}}{\kappa_r\kappa_\nu}\left(s - \frac{1 - e^{-\kappa_r s}}{\kappa_r} - \frac{1 - e^{-\kappa_\nu s}}{\kappa_\nu} + \frac{1 - e^{-(\kappa_r + \kappa_\nu)s}}{\kappa_r + \kappa_\nu}\right),
\end{aligned}$$

$$b(s) = \frac{1 - e^{-\kappa_r s}}{\kappa_r},$$

$$c(s) = \frac{1 - e^{-\kappa_\nu s}}{\kappa_\nu},$$

subject to the transversality condition

$$\begin{aligned}
\bar{r} >& \bar{\nu} + \frac{\sigma_r^2}{2\kappa_r^2} + \frac{\sigma_\nu^2}{2\kappa_\nu^2} - \frac{\sigma_r\sigma_Y\rho_{rY}}{\kappa_r} - \frac{\sigma_r\sigma_\nu\rho_{r\nu}}{\kappa_r\kappa_\nu} + \frac{\sigma_Y\sigma_\nu\rho_{Y\nu}}{\kappa_\nu} \\
&- \sigma_Y\sigma_\Lambda\rho_{Y\Lambda} + \frac{\sigma_r\sigma_\Lambda\rho_{r\Lambda}}{\kappa_r} - \frac{\sigma_\nu\sigma_\Lambda\rho_{\nu\Lambda}}{\kappa_\nu}.
\end{aligned} \tag{28}$$

Proof. See appendix A.6. □

Proposition 6 identifies the Bakshi and Chen [2001] firm valuation model as a special case of the generalized fundamental asset pricing equation (5).[37]

[37] See Bakshi and Chen [2001, equations (9)–(15)]. Note that there are wrong signs in the parameters μ_g and μ_r in Bakshi and Chen [2001, equation (9)].

Here, we have assumed that the company is entirely equity-financed in which case (27) constitutes the firm value. Bakshi and Chen [2001] renounce on this assumption and therefore interpret (27) as the value of the firm's equity. The transversality condition guarantees that the integral in (27) and thus the firm value is finite. If the transversality condition were violated, the integral would not converge and (27) would be ill-defined. The economic interpretation of (27) is straightforward. The integral in (27) gives the value of a perpetuity that pays \$1.00 today and grows according to the earnings process of the firm. The factors in front of the integral scale the value of this perpetuity to match the firm's current dividend level.

The Bakshi and Chen [2001] firm valuation model can be considered as an extension of the Gordon Growth Model that accounts for stochastic interest rates and stochastic earnings growth. It thus includes the basic Gordon [1962] model as a special case.

Corollary 1 *Suppose that the assumptions of proposition 6 hold. Suppose further that* $\sigma_r = \sigma_Y = \sigma_\nu = 0$*,* $r(t) = \bar{r}$*, and* $\nu(t) = \bar{\nu}$*. The solution of* (26) *is then given by*

$$V(Y) = \frac{\alpha Y(t)}{\bar{r} - \bar{\nu}} = \frac{DI(Y,t)}{\bar{r} - \bar{\nu}}, \tag{29}$$

subject to the transversality condition $\bar{r} > \bar{\nu}$.

Proof. Inserting the assumptions $\sigma_r = \sigma_Y = \sigma_\nu = 0$, $r(t) = \bar{r}$, and $\nu(t) = \bar{\nu}$ into (27) and solving the resulting integral directly gives (29). □

Under the assumptions of corollary 1 and $\sigma_Y > 0$, (27) reduces to an extended Gordon Growth Model. The solution corresponds to (29) with the discount rate in the denominator of (29) given by the risk-free rate of interest r plus a risk premium $\sigma_Y \sigma_\Lambda \rho_{Y\Lambda}$. The risk premium accounts for the covariance of the earnings process with the discount factor process, i.e. for the firm's systematic risk.

The approach of Bakshi and Chen [2001] resolves many shortcomings of the classical Gordon Growth Model. Yet, the earnings process proposed by Bakshi and Chen [2001] still cannot deal with negative earnings and the general setup of the model obscures its application to firms that do not pay dividends. The first shortcoming is resolved by Dong [2000] who presents an adaptation of the Bakshi and Chen [2001] model that can deal with negative earnings. The second shortcoming however remains an open issue. Furthermore, the model does not allow for investor or management interactions. Despite these remaining deficiencies, Bakshi and Chen [2001] find an excellent empirical pricing performance of their model.

5 General Equilibrium Models

This section presents five general equilibrium models of firm valuation as special cases of the generalized fundamental asset pricing equation. In contrast to the no-arbitrage models discussed in the previous section, the general equilibrium approaches derive the discount factor endogenously from assumptions about investors' preferences. The section is concerned with the general equilibrium models of Brennan and Schwartz [1982a,b, 1984] and Schwartz and Moon [2000, 2001]. These models build on a Cox, Ingersoll, and Ross [1985b] production economy with agents that display logarithmic utility. We first analyze the models of Brennan and Schwartz [1982a,b, 1984] and then the approaches of Schwartz and Moon [2000, 2001].

5.1 Brennan and Schwartz [1982a,b, 1984]

The underlying economy of the Brennan and Schwartz [1982a,b, 1984] models is given by the production economy of Cox, Ingersoll, and Ross [1985b].[38] In contrast to Cox, Ingersoll, and Ross [1985b], Brennan and Schwartz [1982a,b, 1984] allow the state variables of the economy to follow controlled Itô-Poisson processes and not only uncontrolled Itô processes. They further suppose that investors exhibit logarithmic utility functions

$$u(c, K, t) = e^{-\delta t} \ln c(t), \tag{30}$$

where δ represents the investors' subjective time preference factor. These assumptions directly lead to the following discount factor in the economy.

Proposition 7 *Suppose that the assumptions of Cox, Ingersoll, and Ross [1985b] hold, that the state variables $K(t,m)$ follow controlled Itô-Poisson processes, and that investors have logarithmic utility functions $u(c, K, t) = e^{-\delta t} \ln c(t)$. The stochastic discount factor in equilibrium is then given by*

$$\frac{d\Lambda(K,t,m)}{\Lambda(K,t,m)} = -r(K,t,m)\,dt - \sigma_W(K,t,m)\,dz_W(t). \tag{31}$$

Proof. See appendix A.7. □

Here $\sigma_W(K,t,m)$ denotes the volatility of the return on aggregate wealth in the economy $W(t)$ and $z_W(t)$ the standard Wiener process driving the return on aggregate wealth. Proposition 7 derives the discount factor implied by a Cox, Ingersoll, and Ross [1985b] economy for agents with logarithmic utility functions. The discount factor has only one risk factor, namely the Wiener process driving the return on aggregate wealth. Furthermore, the discount factor process is non-controllable since the agents' preferences uniquely determine the discount factor, i.e. $\mathbb{N} = \emptyset$. Comparing (31) to the discount factor (25) assumed by Bakshi and Chen [2001] yields an interesting result. Interpret σ_Λ and $dz_\Lambda(t)$ in (25) as the volatility of the return on aggregate wealth and the risk factor driving the aggregate wealth process, respectively. The discount factor assumed by Bakshi and Chen [2001] is then consistent with a Cox, Ingersoll, and Ross [1985b] economy if agents display logarithmic utility and the return volatility of aggregate wealth is constant. Thus, under some additional assumptions the

[38] For the detailed assumptions, see Cox, Ingersoll, and Ross [1985b].

no-arbitrage model of Bakshi and Chen [2001] can be interpreted as a general equilibrium model on the basis of a Cox, Ingersoll, and Ross [1985b] economy.

In order to simplify their setup, Brennan and Schwartz [1982a,b, 1984] assume the risk-free rate of interest $r(K,t,m)$ to be constant and the volatility of the return on aggregate wealth $\sigma_W(K,t,m)$ to be non-controllable.[39] Therefore, the discount factor underlying the Brennan and Schwartz [1982a,b, 1984] models is given by[40]

$$\frac{d\Lambda(K,t)}{\Lambda(K,t)} = -r\,dt - \sigma_W(K,t)\,dz_W(t). \tag{32}$$

Having derived the discount factor of the Brennan and Schwartz [1982a,b, 1984] models, we now turn to the specification of the state variables and the cash flow function. Thereby, we take a slightly generalized version of Brennan and Schwartz [1984] as our starting point. This setup allows us to show that the three models of Brennan and Schwartz [1982a,b, 1984] are special cases of an overall setup.

The first state variable in the Brennan and Schwartz [1984] model is the return on assets before interest and taxes $v(t)$ which we assume to follow

$$dv(t) = \mu_v(v,\psi,t)\,dt + \sigma_v(v,\psi,t)\,dz_v(t) + (\hat{v} - v(t))\,dq_v(t), \tag{33}$$

where $\mu_v(v,\psi,t)$ designates the drift, i.e. the expected change, in the return on assets, $\sigma_v(v,\psi,t)$ the volatility of changes in the return on assets, $z_v(t)$ a standard Wiener process, $\hat{v}$ the fixed point jump level, and $q_v(t)$ a Poisson process with intensity $\lambda_v(v)$ and deterministic jump height $\theta_v(t) = 1$. The correlation coefficient between the Wiener process increment $dz_v(t)$ and the risk factor increment $dz_W(t)$ is represented by ρ_{vW}. The Poisson process $q_v(t)$ is assumed to be independent of the risk factor $z_W(t)$. The implication of the last term in (33) is that whenever a Poisson event occurs, the return on assets jumps to the level $\hat{v}$. Our specification of the return on asset process (33) constitutes a slight generalization of the original setup of Brennan and Schwartz [1984]. Specifically, Brennan and Schwartz [1984] model the return on asset process as a pure Itô diffusion without a jump component. Yet, in their earlier papers, Brennan and Schwartz [1982a,b] use the jump-diffusion specification of (33). The adaptation of the return on asset process is our only generalization of the original Brennan and Schwartz [1984] model. The remaining specifications of the state variables are exactly those of Brennan and Schwartz [1984]. Economically, (33) can be interpreted as follows.[41] As long as there is no entry and exit in an industry, the return on assets of a firm evolves continuously as an Itô diffusion which is captured by the first two terms in (33).

[39] Sufficient conditions for a constant risk-free rate of interest are e.g. constant returns and volatilities of the production possibilities in the economy. For alternative assumptions implying a constant risk-free rate of interest, see Brennan and Schwartz [1984, p. 600].

[40] The stochastic discount factor (32) is also consistent with the Intertemporal Capital Asset Pricing Model of Merton [1973a] if investors exhibit logarithmic utility and the risk-free rate of interest is constant.

[41] For an alternative interpretation, see Brennan and Schwartz [1982a,b] who use this setup to model firms in regulated industries.

Yet, whenever the return on assets becomes too low or high, other companies exit or enter the firm's industry driving the firm's return on assets back to a more normal level. This effect can be represented by the Poisson component in (33).

As the second state variable, Brennan and Schwartz [1984] take the book value of assets $BA(t)$ which evolves according to

$$dBA(t) = \psi(t)BA(t)\,dt \qquad\qquad \psi(t) \in [\psi_{min}, \psi_{max}], \tag{34}$$

where $\psi(t)$ denotes the deterministic growth rate in the book value of assets at time t. The growth rate $\psi(t)$ characterizes the firm's investment policy. Brennan and Schwartz [1984] assume that $\psi(t)$ can be chosen endogenously by the firm's management in the exogenously specified interval $[\psi_{min}, \psi_{max}]$.

Brennan and Schwartz [1984] further presume that the company has two classes of corporate securities outstanding, namely common equity and a coupon bond with coupon rate i_D and maturity T. In addition to its investment policy, Brennan and Schwartz [1984] allow the firm to control its financing policy as well. They assume that the face value of outstanding debt $FD(t)$ as the third state variable follows

$$dFD(t) = \zeta(t)FD(t)\,dt \qquad\qquad \zeta(t) \in [\zeta_{min}, \zeta_{max}], \tag{35}$$

where the growth rate in the face value of debt $\zeta(t)$ is chosen endogenously by the firm in the feasible interval $[\zeta_{min}, \zeta_{max}]$. This implies that the firm can optimally manage its capital structure. By choosing a positive $\zeta(t)$, the company issues new debt and thus increases its debt level while a negative $\zeta(t)$ means that the company redeems outstanding debt and thus reduces its debt level. Newly issued debt is supposed to match the maturity T of existing debt. At maturity T, the firm is supposed to buy back the entire outstanding debt. The firm's controls hence include $\psi(t)$ and $\zeta(t)$ with the set of admissible controls given by $\mathbb{M} = [\psi_{min}, \psi_{max}] \times [\zeta_{min}, \zeta_{max}] \subseteq \mathbb{R}^2$. It is assumed that the firm's management acts in the best interest of the company's shareholders. The controls are thus chosen as to maximize the value of the firm's equity.

Finally, Brennan and Schwartz [1984] specify the firm's tax function to be

$$T(t) = \tau\left(v(t)BA(t) - i_D FD(t)\right), \tag{36}$$

where τ denotes the corporate tax rate and $T(t)$ the tax payments at time t. Together, these assumptions imply that the cash flow function for the firm's equity $A_E(v, BA, FD, \psi, \zeta, t)$ is given by

$$\begin{aligned} A_E(v, BA, FD, \psi, \zeta, t) &= DI(v, BA, FD, \psi, \zeta, t) \\ &= \left(v(t) - \psi(t)\right)BA(t) - T(t) - PD(v, BA, FD, \psi, \zeta, t). \end{aligned} \tag{37}$$

The cash flow function for the firm's debt $A_D(v, BA, FD, \psi, \zeta, t)$ is defined as

$$\begin{aligned} A_D(v, BA, FD, \psi, \zeta, t) &= PD(v, BA, FD, \psi, \zeta, t) \\ &= i_D FD(t) - \zeta(t)D(v, BA, FD, t). \end{aligned} \tag{38}$$

Here $PD(v, BA, FD, \psi, \zeta, t)$ stands for the total payout to debt which is composed of the interest payments on outstanding debt and the change in outstanding debt. The market value of the firm's debt $D(v, BA, FD, t)$ appears on the right-hand side of (38) since the firm must issue or redeem its debt at the existing market value. Economically, (37) can be interpreted as follows. The first term on the right-hand side is the difference between earnings before interest and taxes $v(t)BA(t)$ and the firm's capital investment, i.e. the growth in the book value of assets $\psi(t)BA(t)$. This cash flow after investment is further reduced by tax payments and the total cash flow to debt. The remainder is then distributed as dividends. Brennan and Schwartz [1984] thus also belongs to the class of *Dividend Discount Models.* The above formulae show that in determining $\psi(t)$ the firm's management must weigh a higher dividend today against a stronger increase in the book value of assets and possibly higher cash flows and dividends in the future. In choosing the control $\zeta(t)$, a higher current dividend must be balanced against higher future interest payments and an increasing default risk. The cash flow functions to debt and equity themselves however are non-controllable, i.e. $\mathbb{O} = \emptyset$. It is noteworthy that the controls $\psi(t)$ and $\zeta(t)$ do not constitute controls of the cash flow functions. They are simply the original controls of the state variables that also appear in the cash flow functions to debt and equity. In contrast, a controllable cash flow function would require additional control variables given eventually controlled state variables. The above assumptions specify our slightly generalized version of the firm valuation model of Brennan and Schwartz [1984].

Proposition 8 *Suppose that the assumptions of proposition 7 hold, that the risk-free rate of interest r is constant, that the volatility of the return on aggregate wealth is non-controllable, and that the state variable processes are given by* (33)–(35)*. Suppose further that the firm has common equity and a coupon bond outstanding and let the cash flow functions to equity and debt be given by* (37) *and* (38)*, respectively. The value of the firm's equity $E(v, BA, FD, t)$ then satisfies the following partial differential equation*

$$\begin{aligned}\sup_{\psi,\zeta\in\mathbb{M}} \Big[& \frac{1}{2}\sigma_v^2(v,\psi,t)E_{vv} + \left(\mu_v(v,\psi,t) - \sigma_v(v,\psi,t)\sigma_W(K,t)\rho_{vW}\right)E_v \\ & + \lambda_v(v)\left(E(\hat{v}, BA, FD, t) - E(v, BA, FD, t)\right) \\ & + \psi(t)BA(t)E_{BA} + \zeta(t)FD(t)E_{FD} + E_t \\ & + \left(v(t) - \psi(t)\right)BA(t) - T(t) - PD(t) - rE \Big] = 0.\end{aligned} \tag{39}$$

The value of the firm's debt $D(v, BA, FD, t)$ then satisfies the following partial differential equation

$$\begin{aligned}& \frac{1}{2}\sigma_v^2(v,\psi^*,t)D_{vv} + \left(\mu_v(v,\psi^*,t) - \sigma_v(v,\psi^*,t)\sigma_W(K,t)\rho_{vW}\right)D_v \\ & + \lambda_v(v)\left(D(\hat{v}, BA, FD, t) - D(v, BA, FD, t)\right) + \psi^*(t)BA(t)D_{BA} \\ & + \zeta^*(t)FD(t)D_{FD} + D_t + i_D FD(t) - \zeta^*(t)D - rD = 0,\end{aligned} \tag{40}$$

where $\psi^* = \psi^*(v, BA, FD, t)$ and $\zeta^* = \zeta^*(v, BA, FD, t)$ are the optimal controls for which the supremum in (39) *is attained.*

Proof. See appendix A.8. □

Proposition 8 describes a slightly generalized version of the firm valuation model of Brennan and Schwartz [1984] which also allows for discrete jumps in the return on asset process. Again, we have derived this model as a special case of the generalized fundamental asset pricing equation (5). The specialization arises from the assumptions we made about the setup of the economy, the resulting discount factor, the state variable processes, and the cash flow functions of the firm. The PDEs for the equity and debt value of the company must be solved numerically since there do not exist closed-form solutions. The numerical solutions of (39) and (40) depend on the imposed boundary conditions for the debt and equity value. The original Brennan and Schwartz [1984] model is just a special case of proposition 8. It is given in the following corollary.

Corollary 2 (Brennan and Schwartz [1984]) *Suppose that the assumptions of proposition 8 hold. Suppose further that $\lambda_v(v) = 0$. The value of the firm's equity $E(v, BA, FD, t)$ then satisfies the following partial differential equation*

$$\sup_{\psi,\zeta\in\mathbb{M}} \Bigg[\frac{1}{2}\sigma_v^2(v,\psi,t)E_{vv} + \left(\mu_v(v,\psi,t) - \sigma_v(v,\psi,t)\sigma_W(K,t)\rho_{vW}\right)E_v \\ + \psi(t)BA(t)E_{BA} + \zeta(t)FD(t)E_{FD} + E_t \\ + \left(v(t) - \psi(t)\right)BA(t) - T(t) - PD(t) - rE \Bigg] = 0. \tag{41}$$

The value of the firm's debt $D(v, BA, FD, t)$ then satisfies the following partial differential equation

$$\frac{1}{2}\sigma_v^2(v,\psi^*,t)D_{vv} + \left(\mu_v(v,\psi^*,t) - \sigma_v(v,\psi^*,t)\sigma_W(K,t)\rho_{vW}\right)D_v \\ + \psi^*(t)BA(t)D_{BA} + \zeta^*(t)FD(t)D_{FD} + D_t \\ + i_D FD(t) - \zeta^*(t)D - rD = 0, \tag{42}$$

where $\psi^* = \psi^*(v, BA, FD, t)$ and $\zeta^* = \zeta^*(v, BA, FD, t)$ are the optimal controls for which the supremum in (41) is attained.

Proof. Setting $\lambda_v(v) = 0$ in (39) and (40) directly leads to (41) and (42), respectively. □

Corollary 2 identifies the Brennan and Schwartz [1984] firm valuation model as another special case of the generalized fundamental asset pricing equation.[42] The only difference between our general setup and the model of Brennan and Schwartz [1984] is their assumption that the return on assets follows a pure diffusion process. Setting the intensity of the Poisson process $\lambda_v(v) = 0$ in the corollary eliminates the jump component in the general setup making (33) a pure Itô process. In line with the general setup, the PDEs of Brennan and

[42] See Brennan and Schwartz [1984, equations (24)–(25)].

Schwartz [1984] cannot be solved analytically. Using a concrete example, Brennan and Schwartz [1984] impose various boundary conditions on the debt and equity value and solve the model numerically.

The approach of Brennan and Schwartz [1982b] also constitutes a special case of proposition 8. In this paper, Brennan and Schwartz [1982b] simplify the general model in three dimensions. First, they assume that the firm is entirely equity-financed, i.e. $FD(t) = 0$ for all t. This reduces the set of feasible controls for the state variable processes to $\mathbb{M} = [\psi_{min}, \psi_{max}]$ since the company can no longer control its financing policy. Second, the drift and volatility parameters of the discount factor process and the return on asset process are assumed to be time-independent, i.e. $\sigma_W(K,t) = \sigma_W(K)$, $\mu_v(v,\psi,t) = \mu_v(v,\psi)$, and $\sigma_v(v,\psi,t) = \sigma_v(v,\psi)$. Third, they assume that there are no taxes in the economy, i.e. $\tau = 0$.

Corollary 3 (Brennan and Schwartz [1982b]) *Suppose that the assumptions of proposition 8 hold, that the parameters of the discount factor process* (32) *and the state variable process* (33) *are given by* $\sigma_W(K,t) = \sigma_W(K)$, $\mu_v(v,\psi,t) = \mu_v(v,\psi)$, $\sigma_v(v,\psi,t) = \sigma_v(v,\psi)$, *and that there are no taxes in the economy. Suppose further that the firm is entirely equity-financed. The firm value* $V(v, BA)$ *then satisfies the following partial differential equation*

$$\sup_{\psi\in\mathbb{M}} \left[\frac{1}{2}\sigma_v^2(v,\psi)V_{vv} + \left(\mu_v(v,\psi) - \sigma_v(v,\psi)\sigma_W(K)\rho_{vW}\right)V_v \right. \\ + \lambda_v(v)\left(V(\hat{v},BA) - V(v,BA)\right) \\ \left. + \psi(t)BA(t)V_{BA} + \left(v(t) - \psi(t)\right)BA(t) - rV \right] = 0. \tag{43}$$

Proof. Start with the PDE for the firm's equity value in proposition 8. The control $\zeta(t)$ can be dropped since the firm is entirely equity-financed. This leaves us with the supremum operator in (43). Furthermore, without debt in the firm's capital structure, the value of the equity corresponds to the entire firm value, i.e. $E = V$. The partial derivative V_{FD} vanishes, i.e. $V_{FD} = 0$, as it was assumed that $FD(t) = 0$ for all t. The firm value $V(v, BA)$ does not explicitly depend on time t as the firm has no predetermined maturity and all parameters in the discount factor and state variable processes are time-independent. Therefore, the partial derivative of the firm value with respect to t also disappears, i.e. $V_t = 0$. The assumptions of no taxes and no debt further imply that $T(t) = 0$ and $PD(t) = 0$ for all t leaving us with (43). □

Corollary 3 derives the Brennan and Schwartz [1982b] firm valuation model as a special case of the generalized fundamental asset pricing equation.[43] In its general form, the PDE (43) cannot be solved in closed form either. Brennan and Schwartz [1982b, pp. 293–295] present a specialization of their model that lends itself to a partially analytical solution. Specifically, they assume that the return on assets follows $dv(t) = \mu_v\, dt + \sigma_v\, dz_v(t) + (\hat{v} - v(t))\; dq_v(t)$, where the intensity $\lambda_v(v)$ of the Poisson process $q_v(t)$ is constant.

[43] See Brennan and Schwartz [1982b, equation (14)].

The firm valuation model of Brennan and Schwartz [1982a] finally constitutes a straightforward simplification of the above corollary. In contrast to Brennan and Schwartz [1982b], they assume that the growth rate in the book value of assets cannot be chosen endogenously by the firm's management but is exogenously specified as a constant, i.e. $\psi(t) = \psi$.

Corollary 4 (Brennan and Schwartz [1982a]) *Suppose that the assumptions of corollary 3 hold. Suppose further that $\psi(t) = \psi$. The firm value $V(v, BA)$ then satisfies the following partial differential equation*

$$\begin{aligned} &\frac{1}{2}\sigma_v^2(v)V_{vv} + (\mu_v(v) - \sigma_v(v)\sigma_W(K)\rho_{vW})\, V_v \\ &+ \lambda_v(v)\,(V(\hat{v}, BA) - V(v, BA)) \\ &+ \psi BA(t)V_{BA} + (v(t) - \psi)\, BA(t) - rV = 0. \end{aligned} \tag{44}$$

Proof. Since ψ is constant and non-controllable, the set of admissible controls $\mathbb{M}$ is empty. Thus, the supremum operator in (43) vanishes. Setting $\psi(t) = \psi$ in (43) directly gives (44). □

Corollary 4 constitutes the Brennan and Schwartz [1982a] model of firm valuation.[44] Comparable to the Brennan and Schwartz [1982b] approach, there does not exist a closed-form solution of the general PDE (44). Yet, Brennan and Schwartz [1982a, pp. 511–513] are able to obtain analytical solutions in several cases by imposing more restrictive assumptions on the return on asset process, the fixed point jump level $\hat{v}$, and the intensity function of the Poisson process. As in the Bakshi and Chen [2001] approach, the Gordon Growth Model is included in all Brennan and Schwartz [1982a,b, 1984] models as a special case.

Corollary 5 *Suppose that the assumptions of corollary 4 hold. Suppose further that $\mu_v(v) = \sigma_v(v) = 0$ and $\lambda_v(v) = 0$. The solution of (44) is then given by*

$$V(BA) = \frac{(v - \psi)BA(t)}{r - \psi} = \frac{DI(BA)}{r - \psi}, \tag{45}$$

subject to the transversality condition $r > \psi$.

Proof. Inserting the assumptions $\mu_v(v) = \sigma_v(v) = 0$ and $\lambda_v(v) = 0$ into (44) and solving the resulting ordinary differential equation gives (45). □

From an economic point of view, (45) is relevant only for $v > \psi$. Otherwise, the firm value would be zero or negative as the company would never pay positive dividends. In the notation of Brennan and Schwartz, $vBA(t)$ stands for the firm's earnings $Y(t)$ as used in the original Gordon [1962] model. Under the assumptions of corollary 5, the growth rate in earnings ν is given by the growth rate in the book value of assets ψ and the payout rate α by $1 - \psi/v$. Using the alternative assumptions $\mu_v(v) = \mu_v v(t)$ and $\psi = 0$ in corollary 5 also reduces the Brennan and Schwartz [1982a] model to the Gordon Growth Model.

[44] See Brennan and Schwartz [1982a, equation (4)].

Yet, economically it is nonsensical to allow the return on assets $v(t)$ to increase deterministically beyond any bound. For $\mu_v(v) = \mu_v v(t)$, $\sigma_v(v) = \sigma_v v(t)$, and $\lambda_v(v) = \psi = 0$, the Brennan and Schwartz [1982a] model becomes an extended Gordon Growth Model identical to the special case of Bakshi and Chen [2001] discussed in the previous section. However, this setup also suffers from the economically nonsensical assumption that the return on assets can grow beyond any bound.

The Brennan and Schwartz [1982a,b, 1984] models share several advantages over the no-arbitrage approaches proposed by Gordon [1962] and Bakshi and Chen [2001]. First, they derive the stochastic discount factor endogenously from assumptions about investors' preferences. Economically, this is more appealing than an exogenous specification of the discount factor. Second, Brennan and Schwartz [1984] value not only the firm's equity but also derive an expression for the value of the company's debt. Third, the models can deal with negative earnings and the approaches of Brennan and Schwartz [1982a,b] can even handle jumps in the earnings process. Fourth and probably most important, the models of Brennan and Schwartz [1982b, 1984] allow for investor interactions. Namely, they allow the firm to endogenously choose its investment and financing policies. Although the endogenous derivation of the discount factor is advantageous, the restrictive assumptions about investors' preferences, which are necessary for the discount factor to obtain, constitute the main deficiency of the Brennan and Schwartz [1982a,b, 1984] models. Another drawback of the methodologies is their mathematical complexity which permits analytical solutions only for special cases.

5.2 Schwartz and Moon [2000, 2001]

The models of Schwartz and Moon [2000, 2001] are based on the same specification of the Cox, Ingersoll, and Ross [1985b] production economy that underlies Brennan and Schwartz [1982a,b, 1984]. Specifically, Schwartz and Moon [2000, 2001] also assume that investors in a Cox, Ingersoll, and Ross [1985b] economy have logarithmic utility functions, that the risk-free rate of interest is constant, and that the volatility of the return on aggregate wealth is non-controllable. It has already been shown in the preceding section that the discount factor is thus given by

$$\frac{d\Lambda(K,t)}{\Lambda(K,t)} = -r\,dt - \sigma_W(K,t)\,dz_W(t) \tag{32}$$

and the set of admissible controls for the discount factor is empty, i.e. $\mathbb{N} = \emptyset$.

As becomes clear in our derivation of the two models, the approach of Schwartz and Moon [2000] constitutes a special case of Schwartz and Moon [2001]. Therefore, our specifications of the state variable processes and the cash flow function follow the setup of Schwartz and Moon [2001]. The firm's revenues $R(t)$ are the first state variable and follow the Itô process

$$dR(t) = \mu(t)R(t)\,dt + \sigma(t)R(t)\,dz_R(t), \tag{46}$$

where $\mu(t)$ denotes the drift and $\sigma(t)$ the volatility of the revenue growth rate and $z_R(t)$ a standard Wiener process.

The expected revenue growth rate $\mu(t)$ constitutes the second state variable. Its evolution is described by the Ornstein-Uhlenbeck process

$$d\mu(t) = \kappa_\mu \left(\bar{\mu} - \mu(t)\right) dt + \eta(t)\, dz_\mu(t), \tag{47}$$

whereby $\bar{\mu}$ designates the long-term expected growth rate in revenues and κ_μ the mean reversion speed parameter that determines the speed with which $\mu(t)$ reverts back to $\bar{\mu}$. The volatility of changes in $\mu(t)$ is given by $\eta(t)$ and $z_\mu(t)$ is again a standard Wiener process. Since Schwartz and Moon [2000, 2001] are particularly concerned with the valuation of growth companies, the economic intuition behind (47) is the following. In their start-up phase, growth companies display revenue growth rates that are substantially above the long-term mean growth rate $\bar{\mu}$. Yet, over time growth rates come down to the more sustainable level $\bar{\mu}$ which is captured by (47). Furthermore, Schwartz and Moon [2000, 2001] assume that the volatilities $\sigma(t)$ and $\eta(t)$ also revert to lower levels as the company matures. The volatilities are modeled as deterministic mean reversion processes

$$d\sigma(t) = \kappa_\sigma \left(\bar{\sigma} - \sigma(t)\right) dt \tag{48}$$

and

$$d\eta(t) = -\kappa_\eta \eta(t)\, dt. \tag{49}$$

Here $\bar{\sigma}$ represents the long-term volatility of the revenue process while κ_σ and κ_η are the respective mean reversion speed parameters of the two processes. Equation (49) implies that the long-term volatility of the expected revenue growth rate equals zero, i.e. for $t \to \infty$ the stochastic component in (47) disappears.

In modeling the firm's costs $C(t)$, Schwartz and Moon [2000, 2001] only differentiate between variable and fixed costs F whereby variable costs are modeled as a percentage rate $\gamma(t)$ of revenues

$$C(t) = \gamma(t)R(t) + F. \tag{50}$$

The variable cost rate $\gamma(t)$ is taken as the third state variable and assumed to evolve according to an Ornstein-Uhlenbeck process

$$d\gamma(t) = \kappa_\gamma \left(\bar{\gamma} - \gamma(t)\right) dt + \phi(t)\, dz_\gamma(t), \tag{51}$$

whose volatility $\phi(t)$ follows a deterministic mean reversion process

$$d\phi(t) = \kappa_\phi \left(\bar{\phi} - \phi(t)\right) dt. \tag{52}$$

The mean reversion speed parameters are again designated by κ_γ and κ_ϕ and $z_\gamma(t)$ is another standard Wiener process. (51) reflects that growth companies often incur higher costs early in their lives, e.g. due to customer acquisition efforts, that decrease as the firm matures. Additionally, the mean-reverting volatility in (52) captures that the cost structure of a young firm is volatile but stabilizes over time. The correlation coefficients of the Wiener process increments $dz_R(t)$, $dz_\mu(t)$, and $dz_\gamma(t)$ are given by $\rho_{R\mu}$, $\rho_{R\gamma}$, and $\rho_{\mu\gamma}$, respectively. Their correlation coefficients with the risk factor increment $dz_W(t)$ are designated by ρ_{RW}, $\rho_{\mu W}$, and $\rho_{\gamma W}$, respectively.

Schwartz and Moon [2001] determine the firm's depreciation allowances $DE(t)$ as a constant percentage DR of existing plant, property, and equipment $PPE(t)$

$$DE(t) = DR\, PPE(t), \tag{53}$$

and capital expenditures $CE(t)$ as a constant percentage CR of revenues

$$CE(t) = CR\, R(t). \tag{54}$$

These specifications imply that plant, property, and equipment as the fourth state variable obeys the process

$$dPPE(t) = (-DE(t) + CE(t))\ dt. \tag{55}$$

The fifth state variable of the Schwartz and Moon [2001] model is the firm's tax loss carry forward $TL(t)$ which has process representation

$$dTL(t) = \begin{cases} -\left(R(t) - C(t) - DE(t)\right)\ dt & \text{for } R - C - DE - TL \leq 0, \\ -TL(t)\, dt & \text{otherwise.} \end{cases} \tag{56}$$

Economically, the first case of (56) reflects that the existing tax loss carry forward $TL(t)$ is reduced or increased by the firm's pre-tax earnings $R(t) - C(t) - DE(t)$ whenever these fall short of the existing tax loss carry forward. The second part of (56) deals with the opposite case, i.e. when pre-tax earnings exceed the existing tax loss carry forward. In this case, the existing tax loss carry forward is either used up completely making the resulting tax loss carry forward equal to zero or the existing tax loss carry forward remains at zero.

The firm's tax function is defined to be

$$T(t) = \begin{cases} 0 & \text{for } R - C - DE - TL \leq 0, \\ \tau\left(R(t) - C(t) - DE(t) - TL(t)\right) & \text{otherwise.} \end{cases} \tag{57}$$

This tax function implies that the company pays taxes only when its pre-tax earnings $R(t) - C(t) - DE(t)$ exceed the existing tax loss carry forward $TL(t)$. Schwartz and Moon [2001] assume that the firm's free cash flows $R(t) - C(t) - T(t) - CE(t)$ are not distributed as dividends but remain in the firm's cash balance which earns the risk-free rate of interest. The cash balance $X(t)$ as the sixth state variable thus follows

$$dX(t) = (rX(t) + R(t) - C(t) - T(t) - CE(t))\ dt. \tag{58}$$

The above specifications of the six state variable processes do not allow for investor interactions. Consequently, the set of admissible controls for the state variables is empty, i.e. $\mathbb{M} = \emptyset$.

With respect to the cash flow function, Schwartz and Moon [2001] assume that the company does not make any payouts

$$A(R, \mu, \gamma, PPE, TL, X, t) = 0. \tag{59}$$

Thus, the set of controls for the cash flow function is empty as well, i.e. $\mathbb{O} = \emptyset$.

Finally, Schwartz and Moon [2001] impose two boundary conditions on the firm value. First, they assume that at time $T \geq t$ the firm value equals the cash balance $X(T)$ and a multiple of the firm's earnings before interest, taxes, depreciation and amortization (EBITDA)

$$V(R, \mu, \gamma, PPE, TL, X, T) = X(T) + EM(T)\left(R(T) - C(T)\right), \tag{60}$$

where $EM(T)$ represents the EBITDA-multiple applicable to the company at time T. The EBITDA part on the right-hand side of (60) is intended to account for the continuing value of the firm at time T. Since the cash balance at time T represents the accumulated free cash flows up to T and the EBITDA part of (60) should reflect the cash flows available to investors beyond T, the approach of Schwartz and Moon [2001] can be interpreted as a *Discounted Cash Flow Model*. Second, Schwartz and Moon [2001] suppose that the company is liquidated whenever its cash balance falls to a pre-specified level X_L. Its value is then given by

$$V(R, \mu, \gamma, PPE, TL, X_L, t) = S, \tag{61}$$

where S denotes the liquidation value that investors can recoup.[45]

Schwartz and Moon [2001] make no specific assumption with respect to the firm's capital structure. This is unnecessary since the payout policy (59) and the boundary condition (60) imply that their approach is independent of the capital structure. The boundary condition (60) specifies the value of the entire firm at time T and there are no intermediate payouts. Thus, the model always determines the entire firm value irrespective of how this value is distributed among different security classes. Using the two boundary conditions together with the cash flow function and the state variable processes in proposition 4 then provides the Schwartz and Moon [2001] model of firm valuation.

Proposition 9 (Schwartz and Moon [2001]) *Suppose that the assumptions of proposition 7 hold, that the risk-free rate of interest r is constant, that the volatility of the return on aggregate wealth is non-controllable, and that the state variable processes are given by* (46), (47), (51), (55), (56), *and* (58). *Suppose further that the cash flow function is given by* (59) *and the boundary conditions by* (60) *and* (61). *The firm value $V(R, \mu, \gamma, PPE, TL, X, t)$ then satisfies the following set of partial differential equations*

$$\begin{aligned}
&\frac{1}{2}\sigma^2(t)R^2(t)V_{RR} + \frac{1}{2}\eta^2(t)V_{\mu\mu} + \frac{1}{2}\phi^2(t)V_{\gamma\gamma} + \sigma(t)\eta(t)\rho_{R\mu}R(t)V_{R\mu} \\
&+ \sigma(t)\phi(t)\rho_{R\gamma}R(t)V_{R_\gamma} + \eta(t)\phi(t)\rho_{\mu\gamma}V_{\mu\gamma} + \left(\mu(t) - \sigma(t)\sigma_W(K,t)\rho_{RW}\right)R(t)V_R \\
&+ \left(\kappa_\mu\left(\bar{\mu} - \mu(t)\right) - \eta(t)\sigma_W(K,t)\rho_{\mu W}\right)V_\mu \\
&+ \left(\kappa_\gamma\left(\bar{\gamma} - \gamma(t)\right) - \phi(t)\sigma_W(K,t)\rho_{\gamma W}\right)V_\gamma + \left(-DE(t) + CE(t)\right)V_{PPE} \\
&+ TL_i(t)\,V_{TL} + \left(rX(t) + R(t) - C(t) - T_i(t) - CE(t)\right)V_X + V_t - rV = 0 \\
&\hspace{20em} i = 1, 2,
\end{aligned} \tag{62}$$

[45] While Schwartz and Moon [2001] assume S to be zero, we employ here a more general setup.

where

$$TL_1(t) = -(R(t) - C(t) - DE(t)), \quad T_1(t) = 0 \qquad \textit{for } R - C - DE - TL \leq 0,$$
$$TL_2(t) = -TL(t), \quad T_2(t) = \tau\,(R(t) - C(t) - DE(t) - TL(t)) \qquad \textit{otherwise},$$

subject to the boundary conditions

$$V(R, \mu, \gamma, PPE, TL, X_L, t) = S, \tag{63a}$$
$$V(R, \mu, \gamma, PPE, TL, X, T) = X(T) + EM(T)\,(R(T) - C(T)). \tag{63b}$$

Proof. See appendix A.9. □

Proposition 9 identifies the Schwartz and Moon [2001] model for the valuation of growth companies as yet another special case of the generalized fundamental asset pricing equation (5). Specifically, (62) constitutes the set of PDEs for the firm value in the Schwartz and Moon [2001] setup.[46] It is noteworthy that the Schwartz and Moon [2001] model implies two different PDEs for the firm value. The first equation with $TL_1(t)$ and $T_1(t)$ is valid for firms whose current pre-tax earnings $R(t) - C(t) - DE(t)$ are below the existing tax loss carry forward $TL(t)$. Such firms do not pay taxes and their tax loss carry forward evolves according to the first case in (56). All other firms pay taxes according to the second case in the tax function (57) and their tax loss carry forward is zero. In these circumstances, the second PDE with $T_2(t)$ and $TL_2(t)$ specifies the evolution of the firm value. Given the complexity of the above PDEs, it comes as no surprise that both equations cannot be solved in closed form.

The methodology of Schwartz and Moon [2000] is a simpler version of the general Schwartz and Moon [2001] model. In this paper, the variable cost rate is assumed to be constant, i.e. $\gamma(t) = \bar{\gamma}$ and $\phi(t) = \bar{\phi} = 0$, and there are neither depreciation allowances nor capital expenditures, i.e. $DE(t) = CE(t) = 0$. Under these assumptions, proposition 9 reduces to the Schwartz and Moon [2000] model of firm valuation.

Corollary 6 (Schwartz and Moon [2000]) *Suppose that the assumptions of proposition 9 hold. Suppose further that $\gamma(t) = \bar{\gamma}$, $\phi(t) = \bar{\phi} = 0$, and $DE(t) = CE(t) = 0$. The firm value $V(R, \mu, TL, X, t)$ then satisfies the following set of partial differential equations*

$$\begin{aligned}
&\frac{1}{2}\sigma^2(t)R^2(t)V_{RR} + \frac{1}{2}\eta^2(t)V_{\mu\mu} + \sigma(t)\eta(t)\rho_{R\mu}R(t)V_{R\mu} \\
&+ (\mu(t) - \sigma(t)\sigma_W(K,t)\rho_{RW})\,R(t)V_R \\
&+ (\kappa_\mu(\bar{\mu} - \mu(t)) - \eta(t)\sigma_W(K,t)\rho_{\mu W})\,V_\mu + TL_i(t)\,V_{TL} \\
&+ (rX(t) + R(t) - C(t) - T_i(t))\,V_X + V_t - rV = 0 \qquad i = 1, 2,
\end{aligned} \tag{64}$$

[46] Schwartz and Moon [2001] do not derive these PDEs in their original article as they directly focus on a numerical solution of the model via simulation. It can be shown that the firm values obtained from a numerical solution of the model indeed satisfy the above PDEs.

where

$$TL_1(t) = -\left(R(t) - C(t)\right), \quad T_1(t) = 0 \qquad \textit{for } R - C - TL \leq 0,$$
$$TL_2(t) = -TL(t), \quad T_2(t) = \tau\left(R(t) - C(t) - TL(t)\right) \qquad \textit{otherwise},$$

subject to the boundary conditions

$$V(R, \mu, TL, X_L, t) = S, \tag{65a}$$
$$V(R, \mu, TL, X, T) = X(T) + EM(T)\left(R(T) - C(T)\right). \tag{65b}$$

Proof. Inserting $\gamma(t) = \bar{\gamma}$, $\phi(t) = \bar{\phi} = 0$, and $DE(t) = CE(t) = 0$ into (62) directly yields (64). □

Corollary 6 shows that the more restrictive assumptions of Schwartz and Moon [2000] decrease the complexity of the Schwartz and Moon [2001] setup substantially. The firm value is reduced to a function of four state variables and time t. The variable cost rate $\gamma(t)$ and the level of plant, property, and equipment $PPE(t)$ are no longer relevant state variables of the model. Furthermore, the approach of Schwartz and Moon [2000] is only a two-factor model since the variable cost rate is assumed to be constant while it follows an Ornstein-Uhlenbeck process in the more general setup. Yet, despite these simplifications there does not exist an analytical solution of the PDEs (64). Thus, the Schwartz and Moon [2000] model also needs to be solved numerically.

Due to the terminal boundary condition at time T, the bankruptcy condition, and the assumption of no intermediate payouts, the Schwartz and Moon [2000, 2001] models cannot be directly related to the *Dividend Discount Models* of Gordon [1962], Bakshi and Chen [2001], and Brennan and Schwartz [1982a,b, 1984]. Yet, if we are willing to change these assumptions, the Schwartz and Moon [2000] approach can be related to the other models. In particular, drop the two boundary conditions and assume that the firm's free cash flows are distributed as dividends instead of being accumulated in the company's cash account. Assume further that the volatility parameters $\sigma(t)$ and $\eta(t)$ are constant and let F, $TL(t)$, τ, and $X(t)$ all be equal to zero. It is straightforward to demonstrate that under these assumptions the Schwartz and Moon [2000] model reduces to the Bakshi and Chen [2001] model with a constant risk-free rate of interest and the diffusion term in the discount factor given by the diffusion term that drives the return on aggregate wealth. It has already been shown in section 4.2 that the Bakshi and Chen [2001] approach in turn comprises the Gordon Growth Model as a special case. Consequently, the Schwartz and Moon [2000] model can also be reduced to the Gordon [1962] model. Namely, presume that in addition to the previous assumptions the expected growth rate in revenues $\mu(t)$ is constant and the volatility parameters equal zero, i.e. $\sigma(t) = \bar{\sigma} = \eta(t) = 0$. It is easy to verify that in this case the Schwartz and Moon [2000] methodology simplifies to the standard Gordon Growth Model.

The key advantage of the models of Schwartz and Moon [2000, 2001] is their detailed modeling of six state variables which provides many degrees of freedom when adapting the models to a specific firm. In particular, it allows the models to capture the typical characteristics of growth companies, as e.g. a time-varying expected growth rate in revenues and a time-varying volatility in revenue growth. Furthermore, since the models of Schwartz and Moon [2000, 2001] are not dividend-based, they are also applicable to firms that do not pay dividends. This contrasts with the approaches of Gordon [1962], Bakshi and Chen [2001], and Brennan and Schwartz [1982a,b, 1984] which can only deal with dividend-paying firms. The main shortcoming of the Schwartz and Moon [2000, 2001] methodologies is the missing incorporation of investor interactions. This constitutes a step backwards with respect to the models of Brennan and Schwartz [1982b, 1984]. Obviously, the restrictive assumptions that underlie the stochastic discount factor and the models' complexity which necessitates numerical solution approaches constitute further deficiencies.

6 Summary and Conclusion

This paper has developed a generalized SDF framework of firm valuation that includes all existing firm valuation models as special cases. We have generalized the SDF framework of asset pricing in continuous time by introducing controlled state variable, discount factor, and cash flow processes. The fundamental asset pricing equation implied by the generalized SDF framework constitutes the basis of all asset pricing and firm valuation models in continuous time. By restricting the classes of admissible state variable and discount factor processes, we have been able to transform the generalized fundamental asset pricing equation into an HJB equation which can be solved using stochastic control theory. We have then derived the firm valuation models of Gordon [1962], Bakshi and Chen [2001], Brennan and Schwartz [1982a,b, 1984], and Schwartz and Moon [2000, 2001] as special cases of the generalized fundamental asset pricing equation and related the approaches to each other.

In our analysis, we have only dealt with the SDF framework in continuous-time. We have left open the generalization of the discrete-time SDF framework to controlled state variable, discount factor, and cash flow processes. The discrete-time counterpart to our setup however can be developed in direct analogy to the continuous-time framework. The solution approach via stochastic control theory which in discrete time is called stochastic dynamic programming also remains valid. It is then straightforward to show that the firm valuation models in discrete time, as e.g. Campbell and Shiller [1987, 1988], Berk, Green, and Naik [1999], Lee, Myers, and Swaminathan [1999], Ang and Liu [2001], and Bekaert and Grenadier [2001], represent special cases of the generalized fundamental asset pricing equation in discrete time.

In order to reduce the generalized fundamental asset pricing equation to a standard HJB equation, we have confined ourselves to Itô-Poisson processes for the state variables and Itô processes for the discount factor. In general, it is perfectly feasible to work with less restrictive process assumptions. Yet, as there exist hardly any asset pricing models and no single firm valuation model which employ more general process specifications, as e.g. Lévy processes, we have limited our attention to the above process classes. These cover the vast majority of existing asset pricing and firm valuation models while at the same time keeping the mathematical complexity low.

Finally, we have derived only those firm valuation models as special cases of the generalized fundamental asset pricing equation that value the entire firm and that apply to a wide range of companies. We have not demonstrated how contingent claims and real options models that deal with the valuation of specific corporate securities or specific types of firms can be deduced as special cases of the generalized fundamental asset pricing equation. Their derivation however is straightforward. It proceeds exactly along the same lines as the expositions in sections 4 and 5. One simply imposes the respective model's assumptions concerning the state variable processes, the discount factor, the cash flow function, the admissible control policies, etc. on one of the fundamental PDEs (11) or (14) and solves the resulting equation.

With respect to the existing asset pricing and firm valuation literature, the generalized SDF framework of firm valuation developed in this paper displays several advantages. First, the introduction of controlled state variable, discount factor, and cash flow processes enables us to deal with investor interactions and incomplete market setups in a unified SDF framework which is not possible in the standard SDF notation. Second, the generalized SDF framework can handle firm valuation models from the contingent claims, the real options, and the asset pricing literature in a consistent manner. Third, it provides an ideal framework to formalize qualitative notions, as e.g. control premia, that are difficult to capture in existing models. Fourth and most importantly, it highlights that all firm valuation models can differ in only 6 dimensions: state variables, SDF derivation, SDF specification, cash flow function, set of feasible control policies, and applicable boundary conditions.

A Appendix

A.1 Proof of Proposition 1

Proof. Assume that the current state of the system is given by $(K,t) \in \mathbb{R}^k \times [0,\infty)$. Let $t_S < \infty$ be a stopping time with $t_S \geq t$ and $M'(K,t)$, $N'(K,t)$, $O'(K,t)$ be arbitrary admissible control policies. Then the performance function $Q(K,t,M',N',O')$ can be written as

$$\begin{aligned} Q(K,t,M',N',O') &= E_t^P\left[\int_t^{\infty} \frac{\Lambda(K,s,M',N')}{\Lambda(K,t,M',N')} A(K,s,M',O')\,ds\right] \\ &= E_t^P\left[\int_t^{t_S} \frac{\Lambda(K,s,M',N')}{\Lambda(K,t,M',N')} A(K,s,M',O')\,ds \right. \\ &\qquad \left. + \frac{\Lambda(K,t_S,M',N')}{\Lambda(K,t,M',N')} Q(K,t_S,M',N',O')\right], \end{aligned} \tag{66}$$

where $Q(K,t_S,M',N',O')$ denotes the performance function under the policies $M'(K,t)$, $N'(K,t)$, and $O'(K,t)$ starting in state (K,t_S).[47] (66) simply states that today's value of any controllable asset under control policies $M'(K,t)$, $N'(K,t)$, and $O'(K,t)$ can be decomposed into the value of the cash flows up to the stopping time t_S and the value of the asset at the stopping time $Q(K,t_S,M',N',O')$.

Now let $L = \{(a,b) \in \mathbb{R}^k \times [0,\infty) : b < t'\}$ and define t_S to be the first exit of $K(t)$ from L. Suppose the control laws $M'(K,t)$, $N'(K,t)$, and $O'(K,t)$ to be

$$M'(K,t) = \begin{cases} m & \text{if } (K,t) \in L, \\ M^*(K,t) & \text{if } (K,t) \notin L, \end{cases} \tag{67}$$

$$N'(K,t) = \begin{cases} n & \text{if } (K,t) \in L, \\ N^*(K,t) & \text{if } (K,t) \notin L, \end{cases} \tag{68}$$

$$O'(K,t) = \begin{cases} o & \text{if } (K,t) \in L, \\ O^*(K,t) & \text{if } (K,t) \notin L, \end{cases} \tag{69}$$

where $m \in \mathbb{M}(K,t)$, $n \in \mathbb{N}(K,t)$, and $o \in \mathbb{O}(K,t)$ are arbitrary. The control laws $M'(K,t)$, $N'(K,t)$, and $O'(K,t)$ imply that one applies arbitrary controls up to the stopping time t_S and then switches to the optimal control laws $M^*(K,t)$, $N^*(K,t)$, and $O^*(K,t)$. Then it must hold

$$\begin{aligned} &V(K,t) \geq Q(K,t,M',N',O') \\ &= E_t^P\left[\int_t^{t_S} \frac{\Lambda(K,s,M',N')}{\Lambda(K,t,M',N')} A(K,s,M',O')\,ds + \frac{\Lambda(K,t_S,M',N')}{\Lambda(K,t,M',N')} V(K,t_S)\right]. \end{aligned} \tag{70}$$

[47] See Krylov [1980].

Defining a small time interval Δt and letting $t' \downarrow t + \Delta t$ yields

$$\begin{aligned} V(K,t) &\geq Q(K,t,M',N',O') \\ &\approx E_t^P \left[A(K,t,M',O')\,\Delta t + \frac{\Lambda(K,t+\Delta t,M',N')}{\Lambda(K,t,M',N')} V(K,t+\Delta t) \right]. \end{aligned} \tag{71}$$

The right-hand side of (71) can be rewritten as

$$E_t^P \left[A(K,t,M',O')\,\Delta t + \left(1 + \frac{\Delta\Lambda(K,t,M',N')}{\Lambda(K,t,M',N')}\right) (V(K,t) + \Delta V(K,t)) \right]. \tag{72}$$

Combining (71) and (72) gives

$$\begin{aligned} E_t^P \Bigg[A(K,t,M',O')\,\Delta t &+ \frac{\Delta\Lambda(K,t,M',N')}{\Lambda(K,t,M',N')} V(K,t) \\ &+ \frac{\Delta\Lambda(K,t,M',N')}{\Lambda(K,t,M',N')} \Delta V(K,t) + \Delta V(K,t) \Bigg] \leq 0. \end{aligned} \tag{73}$$

Taking the limit of (73) as $\Delta t \to 0$, collecting terms, and recalling that $M'(K,t) = m$, $N'(K,t) = n$, and $O'(K,t) = o$ yields

$$E_t^P \left[A(K,t,m,o)\,dt + \frac{1}{\Lambda(K,t,m,n)} d(\Lambda(K,t,m,n)V(K,t)) \right] \leq 0. \tag{74}$$

Since the control laws $M'(K,t)$, $N'(K,t)$, and $O'(K,t)$ for $(K,t) \in L$ are arbitrary, (74) holds for all choices of $m \in \mathbb{M}(K,t)$, $n \in \mathbb{N}(K,t)$, $o \in \mathbb{O}(K,t)$. The supremum of (74) and thus equality in (74) is attained for $m = M^*(K,t)$, $n = N^*(K,t)$, and $o = O^*(K,t)$, i.e. for the optimal control policies. This is easiest seen from (71). Thus, it holds

$$\sup_{m,n,o} E_t^P \left[A(K,t,m,o)\,dt + \frac{1}{\Lambda(K,t,m,n)} d(\Lambda(K,t,m,n)V(K,t)) \right] = 0, \tag{75}$$

where the supremum is realized for $m = M^*(K,t)$, $n = N^*(K,t)$, and $o = O^*(K,t)$. Since we chose the point (K,t) arbitrary, equation (75) holds for all $(K,t) \in \mathbb{R}^k \times [0,\infty)$. □

A.2 Proof of Proposition 2

Proof. Applying Itô's lemma on the second term in (75) yields

$$\begin{aligned} \sup_{m,n,o} \Bigg[A(K,t,m,o)\,dt &+ E_t^P \left[\frac{d\Lambda(K,t,m,n)}{\Lambda(K,t,m,n)} \right] V(K,t) \\ &+ E_t^P[dV(K,t)] + E_t^P \left[\frac{d\Lambda(K,t,m,n)}{\Lambda(K,t,m,n)} dV(K,t) \right] \Bigg] = 0. \end{aligned} \tag{76}$$

The existence of a risk-less asset ensures that[48]

$$E_t^P \left[\frac{d\Lambda(K,t,m,n)}{\Lambda(K,t,m,n)} \right] = -r(K,t,m)\,dt, \tag{77}$$

[48] See e.g. Cochrane [2001, p. 31].

where $r(K,t,m)$ denotes the instantaneous risk-free rate of interest. Using (77) in (76) and rearranging directly gives

$$\begin{aligned}\sup_{m,n,o} \Bigg[& A(K,t,m,o)\,dt + E_t^P[dV(K,t)] \\ & - r(K,t,m)V(K,t)\,dt + E_t^P\left[\frac{d\Lambda(K,t,m,n)}{\Lambda(K,t,m,n)}\,dV(K,t)\right]\Bigg] = 0 \\ & \qquad \forall (K,t) \in \mathbb{R}^k \times [0,\infty).\end{aligned} \tag{78}$$

□

A.3 Proof of Proposition 3

Proof. Applying Itô's lemma for Itô-Poisson processes on $dV(K,t)$ and using the specification of the state variable process (9) yields[49]

$$\begin{aligned} dV(K,t) =& \mu_K^T(K,t,m)V_K(K,t)\,dt + V_K^T(K,t)\Sigma_K(K,t,m)\,dz_K(t) \\ & + \lambda_K^T(K,t,m)\,E_t^P\left[(V(K+v_{\cdot i}\theta_i,t) - V(K,t))_{i=1}^y\right]\,dt \\ & + \frac{1}{2}\,\mathrm{tr}\left[\Sigma_K(K,t,m)\Sigma_K^T(K,t,m)V_{KK}(K,t)\right]\,dt \\ & + V_t(K,t)\,dt. \end{aligned} \tag{79}$$

Thus, the second term in (8) is given by

$$\begin{aligned} E_t^P[dV(K,t)] =& \mu_K^T(K,t,m)V_K(K,t)\,dt \\ & + \lambda_K^T(K,t,m)\,E_t^P\left[(V(K+v_{\cdot i}\theta_i,t) - V(K,t))_{i=1}^y\right]\,dt \\ & + \frac{1}{2}\,\mathrm{tr}\left[\Sigma_K(K,t,m)\Sigma_K^T(K,t,m)V_{KK}(K,t)\right]\,dt \\ & + V_t(K,t)\,dt. \end{aligned} \tag{80}$$

Next, we compute the last term of (8). Thereby, we only need to account for the diffusion terms of the discount factor process (10) and the asset value process (79) since all other terms lead to products of order higher than dt. Thus, it holds

$$\begin{aligned} & E_t^P\left[\frac{d\Lambda(K,t,m,n)}{\Lambda(K,t,m,n)}\,dV(K,t)\right] \\ & = E_t^P\left[-\sigma_\Lambda^T(K,t,m,n)\,dz_\Lambda(t)V_K^T(K,t)\Sigma_K(K,t,m)\,dz_K(t)\right] \\ & = -\sigma_\Lambda^T(K,t,m,n)\rho_{K\Lambda}^T\Sigma_K^T(K,t,m)V_K(K,t)\,dt. \end{aligned} \tag{81}$$

Inserting (80) and (81) into (8), dividing by dt, and using the definition of the partial differential operator $L^{m,n,o}[F(K,t)]$ as given in (12) leads to (11). □

[49] Itô's lemma for Itô-Poisson processes is given e.g. in Merton [1990], Dixit and Pindyck [1994], and Duffie [2001].

A.4 Proof of Proposition 4

Proof. Instead of using matrix notation, (11) can also be written as

$$\begin{aligned}
&\sup_{m,n,o}\Bigg[A(K,t,m,o) + L^{m,n,o}[V(K,t)] - r(K,t,m)V(K,t) \\
&\quad - \sum_{h=1}^{k}\sum_{i=1}^{x_K}\sum_{j=1}^{x_\Lambda} \Sigma_{K_{hi}}(K,t,m)\sigma_{\Lambda_j}(K,t,m,n)\rho_{K\Lambda_{ij}} V_{K_h}(K,t)\Bigg] \\
&\quad + V_t(K,t) = 0 \qquad\qquad \forall (K,t) \in \mathbb{R}^k \times [0,\infty),
\end{aligned} \tag{82}$$

where $L^{m,n,o}[F(K,t)]$ designates the partial differential operator

$$\begin{aligned}
L^{m,n,o}[F(K,t)] =& \sum_{i=1}^{k} \mu_{K_i}(K,t,m)\, F_{K_i}(K,t) \\
&+ \sum_{i=1}^{y} \lambda_i(K,t,m)\, E_t^P\left[F(K + \upsilon_{\cdot i}\theta_i, t) - F(K,t)\right] \\
&+ \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} \left(\Sigma_K(K,t,m)\Sigma_K^T(K,t,m)\right)_{ij} F_{K_iK_j}(K,t).
\end{aligned} \tag{83}$$

Here $\Sigma_{K_{hi}}(K,t,m)$ is the entry of the volatility matrix $\Sigma_K(K,t,m)$ at position hi. The same notation applies to $\rho_{K\Lambda_{ij}}$, $\sigma_{\Lambda_j}(K,t,m,n)$, $\mu_{K_i}(K,t,m)$, and $\lambda_i(K,t,m)$. $V_{K_h}(K,t)$ denotes the partial derivative of $V(K,t)$ with respect to the state variable $K_h(t)$. The assumptions of proposition 4 imply that $\Sigma_{K_{hi}}(K,t,m) = \sigma_{K_h}(K,t,m)$ for $h = i$ and zero otherwise and $\big(\Sigma_K(K,t,m)\,\Sigma_K^T(K,t,m)\big)_{ij} = \sigma_{K_i}(K,t,m)\sigma_{K_j}(K,t,m)\rho_{K_iK_j}$. Furthermore, according to the assumptions $x_K = k$ and $y = k$. Finally, given a diagonal matrix $\Upsilon_K(K,t,m)$, $\upsilon_{\cdot i}(K,t,m) = \iota_i \upsilon_{K_i}(K,t,m)$ where ι_i describes a k-dimensional vector with a 1 in the ith row and zeroes in all other positions. Inserting these results into the above equations gives (14) with partial differential operator (15). □

A.5 Proof of Proposition 5

Proof. Using the definition of the partial differential operator (15) and the specification of the state variable process (16), $L[V(Y,t)]$ is given by

$$L[V(Y,t)] = (1-\alpha)\beta Y(t) V_Y(Y,t). \tag{84}$$

Inserting (84) and the specification of $A(Y)$, i.e. (17), into proposition 4 leads to (19). Thereby, the following has to be taken into account. First, the last term inside the supremum operator of (14) vanishes as the state variable process is deterministic. Second, the supremum operator disappears since the sets of admissible controls are empty. Third, the partial derivative with respect to time, $V_t(Y,t)$, is equal to zero as the firm value does not explicitly depend on time t. This is due to the fact that the company is assumed to operate forever and that all parameters of the model are time-independent. Consequently, we can also drop time t as a functional argument of the firm value leaving us with (19). It is then straightforward to verify that (20) indeed solves (19). □

A.6 Proof of Proposition 6

Proof. Given the specifications of the state variable processes, $L[V(r, Y, \nu)]$ becomes

$$\begin{aligned} L[V(r, Y, \nu)] =& \kappa_r(\bar{r} - r(t))V_r + \nu(t)Y(t)V_Y + \kappa_\nu(\bar{\nu} - \nu(t))V_\nu \\ &+ \frac{1}{2}\sigma_r^2 V_{rr} + \frac{1}{2}\sigma_Y^2 Y^2(t)V_{YY} + \frac{1}{2}\sigma_\nu^2 V_{\nu\nu} \\ &+ \sigma_r\sigma_Y\rho_{rY}Y(t)V_{rY} + \sigma_r\sigma_\nu\rho_{r\nu}V_{r\nu} + \sigma_Y\sigma_\nu\rho_{Y\nu}Y(t)V_{Y\nu}. \end{aligned} \tag{85}$$

Inserting (85), the cash flow function (24), and the parameters of the discount factor and the state variable processes in the fundamental PDE (14) yields

$$\begin{aligned} &\alpha Y(t) + \kappa_r(\bar{r} - r(t))V_r + \nu(t)Y(t)V_Y + \kappa_\nu(\bar{\nu} - \nu(t))V_\nu \\ &+ \frac{1}{2}\sigma_r^2 V_{rr} + \frac{1}{2}\sigma_Y^2 Y^2(t)V_{YY} + \frac{1}{2}\sigma_\nu^2 V_{\nu\nu} + \sigma_r\sigma_Y\rho_{rY}Y(t)V_{rY} \\ &+ \sigma_r\sigma_\nu\rho_{r\nu}V_{r\nu} + \sigma_Y\sigma_\nu\rho_{Y\nu}Y(t)V_{Y\nu} - r(t)V - \sigma_r\sigma_\Lambda\rho_{r\Lambda}V_r \\ &- \sigma_Y\sigma_\Lambda\rho_{Y\Lambda}Y(t)V_Y - \sigma_\nu\sigma_\Lambda\rho_{\nu\Lambda}V_\nu = 0. \end{aligned} \tag{86}$$

In (86), we have dropped the supremum operator as the sets of admissible controls are empty. The last three terms on the left-hand side follow from imposing the discount factor and state variable specifications on the last term inside the supremum operator in (14). Furthermore, $V(r, Y, \nu)$ does not explicitly depend on calendar time as the firm operates forever and the model parameters are time-independent. Thus, the partial derivative with respect to t equals zero. Further, note that the increment $dW(t)$ does not appear in (86). The cash flow term $A(K, t)$ in (14) is the expected cash flow at time t. Since $W(t)$ is assumed to be a martingale, $E_t^P[dW(t)] = 0$. Therefore, $dW(t)$ does not emerge in (26). Rearranging (86) then leads to (26). Bakshi and Chen [2001] show that the solution of (26) is given by (27). □

A.7 Proof of Proposition 7

Proof. We first derive several intermediate results that we need in the proof. The equilibrium in the production economy of Cox, Ingersoll, and Ross [1985b] is given by the solution of the investors' consumption and portfolio selection problem. The solution is characterized by the derived utility of wealth function $J(W, K, t)$, where $W(t)$ denotes aggregate wealth in the economy. The derived utility of wealth function describes the maximum expected lifetime utility that an investor can attain by pursuing optimal consumption and portfolio rules. Cox, Ingersoll, and Ross [1985b, p. 372] show that the instantaneous risk-free rate of interest in equilibrium is given by

$$\begin{aligned} r(W, K, t, m) =& w^{*T}(W, K, t, m)\mu_P(K, t, m) \\ &+ \frac{J_{WW}(W, K, t)}{J_W(W, K, t)} \frac{\mathrm{var}_t[dW(t)]}{W(t)} \\ &+ \sum_{i=1}^{k} \frac{J_{WK_i}(W, K, t)}{J_W(W, K, t)} \frac{\mathrm{cov}_t[dW(t), dK_i(t, m)]}{W(t)}, \end{aligned} \tag{87}$$

where $w^*(W,K,t,m)$ denotes the vector of the individuals' optimal portfolio weights, $\mu_P(K,t,m)$ the drift, i.e. the expected return vector, of the production possibilities in the economy, and $\text{var}_t[\cdot]$ and $\text{cov}_t[\cdot]$ the variance and covariance operators at time t. Subscripts on $J(W,K,t)$ represent partial derivatives.

Furthermore, Cox, Ingersoll, and Ross [1985b, p. 373] demonstrate that in equilibrium aggregate wealth $W(t)$ evolves according to

$$\begin{aligned} dW(t) = &\left(w^{*T}(W,K,t,m)\mu_P(K,t,m)W(t) - c^*(W,K,t,m)\right) dt \\ &+ w^{*T}(W,K,t,m)\Sigma_P(K,t,m)W(t)\, dz_K(t) \\ =&\mu_W(K,t,m)W(t)\, dt + \sigma_W(K,t,m)W(t)\, dz_W(t). \end{aligned} \tag{88}$$

Here $c^*(W,K,t,m)$ designates the optimal consumption rate at time t and $\Sigma_P(K,t,m)$ the return volatility matrix of the production possibilities in the economy. In the last line, $\mu_W(K,t,m)$ represents the drift, i.e. the expected return, of the aggregate wealth process, $\sigma_W(K,t,m)$ the volatility of the return on aggregate wealth, and $z_W(t)$ a standard Wiener process driving the aggregate wealth process.

It is a well-known result that for investors with logarithmic utility functions the derived utility of wealth function $J(W,K,t)$ has the following functional form[50]

$$J(W,K,t) = \frac{e^{-\delta t} - e^{-\delta T}}{\delta} \ln W(t) + F(K,t,m). \tag{89}$$

Thereby, T designates the investor's time of death and $F(K,t,m)$ a function of the state variables $K(t,m)$ and time t.

Using (88) and (89) in (87) gives the equilibrium risk-free rate of interest in a Cox, Ingersoll, and Ross [1985b] economy for investors with logarithmic utility functions

$$\begin{aligned} r(K,t,m) =&w^{*T}\mu_P - \frac{\left(e^{-\delta t} - e^{-\delta T}\right)/\delta W^2(t)}{\left(e^{-\delta t} - e^{-\delta T}\right)/\delta W(t)} \frac{w^{*T}\Sigma_P\Sigma_P^T w^* W^2(t)}{W(t)} \\ =&w^{*T}\mu_P - w^{*T}\Sigma_P\Sigma_P^T w^*. \end{aligned} \tag{90}$$

Here, we have used the fact that the last term in (87) vanishes since $J_{WK_i} = 0$ for all $i \in \{1,\ldots,k\}$ according to (89).

Finally, for the investors' optimal consumption and portfolio rules, the following envelope condition must hold[51]

$$u_c(c^*,K,t,m) = J_W(W,K,t). \tag{91}$$

Inserting (30) and (89) into (91) and rearranging gives the optimal consumption rule under logarithmic utility as

$$c^*(W,t) = \frac{\delta}{1 - e^{-\delta(T-t)}} W(t). \tag{92}$$

[50] See e.g. Merton [1971, p. 403] or Cox, Ingersoll, and Ross [1985a, p. 389].

[51] See e.g. Merton [1971, p. 381] or Cox, Ingersoll, and Ross [1985b, p. 370].

Now we are ready to derive the discount factor. We have already shown in (4) that the stochastic discount factor in a general equilibrium model is given by

$$\Lambda(K,t,m) = u_c(c^*, K, t, m). \tag{93}$$

Using (93) together with the envelope condition (91), applying Itô's lemma, and dividing by $\Lambda(K,t,m)$ gives the discount factor process as

$$\begin{aligned}\frac{d\Lambda(K,t,m)}{\Lambda(K,t,m)} =& \frac{J_{WW}}{J_W}\,dW(t) + \frac{1}{2}\,\frac{J_{WWW}}{J_W}\,dW^2(t) + \frac{J_{WK}^T}{J_W}\,dK(t,m) \\ &+ \frac{1}{2}\,dK^T(t,m)\frac{J_{WKK}}{J_W}\,dK(t,m) \\ &+ \frac{J_{WWK}^T}{J_W}\,dW(t)\,dK(t,m) + \frac{J_{Wt}}{J_W}\,dt.\end{aligned} \tag{94}$$

According to (89), the partial derivatives of the derived utility of wealth function for investors with logarithmic utility are given by

$$\begin{aligned} J_W =& \frac{e^{-\delta t} - e^{-\delta T}}{\delta W(t)}, & J_{WW} =& -\frac{e^{-\delta t} - e^{-\delta T}}{\delta W^2(t)}, \\ J_{WWW} =& \frac{2\left(e^{-\delta t} - e^{-\delta T}\right)}{\delta W^3(t)}, & J_{WK} =& J_{WKK} = J_{WWK} = 0, \\ J_{Wt} =& -\frac{e^{-\delta t}}{W(t)}. \end{aligned} \tag{95}$$

Using (88), (92), and (95) in (94) yields

$$\begin{aligned}\frac{d\Lambda(K,t,m)}{\Lambda(K,t,m)} =& -\frac{1}{W}\left(\left(w^{*T}\mu_P W - \frac{\delta W}{1-e^{-\delta(T-t)}}\right)dt + w^{*T}\Sigma_P W\,dz_K(t)\right) \\ &+ \frac{1}{2}\,\frac{2}{W^2}\,w^{*T}\Sigma_P\Sigma_P^T w^* W^2\,dt - \frac{\delta}{1-e^{-\delta(T-t)}}\,dt \\ =& -\left(w^{*T}\mu_P - w^{*T}\Sigma_P\Sigma_P^T w^*\right)\,dt - w^{*T}\Sigma_P\,dz_K(t).\end{aligned} \tag{96}$$

Comparing the first term of (96) with (90) shows that the drift of the discount factor process equals the risk-free rate of interest $r(K,t,m)$. Inspection of the second term of (96) and (88) shows that the diffusion term of the discount factor is given by the diffusion term of the return on aggregate wealth. Thus, we can rewrite the discount factor process as

$$\frac{d\Lambda(K,t,m)}{\Lambda(K,t,m)} = -r(K,t,m)\,dt - \sigma_W(K,t,m)\,dz_W(t). \tag{97}$$

□

A.8 Proof of Proposition 8

Proof. We first derive the PDE for the firm's equity (39). The assumed state variable processes imply that

$$\begin{aligned} L^{\psi,\zeta}[E(v, BA, FD, t)] = \\ \mu_v(v, \psi, t)E_v + \psi(t)BA(t)E_{BA} + \zeta(t)FD(t)E_{FD} \\ + \lambda_v(v)\left(E(\hat{v}, BA, FD, t) - E(v, BA, FD, t)\right) + \frac{1}{2}\sigma_v^2(v, \psi, t)E_{vv}. \end{aligned} \tag{98}$$

The assumptions further imply that the discount factor in the economy is given by (32). Inserting (98) together with the stochastic discount factor (32), the state variable processes, and the cash flow function to equity (37) into proposition 4 yields

$$\begin{aligned} \sup_{\psi,\zeta\in\mathbb{M}} \Big[&(v(t) - \psi(t))\, BA(t) - T(t) - PD(t) + \mu_v(v, \psi, t)E_v + \psi(t)BA(t)E_{BA} \\ &+ \zeta(t)FD(t)E_{FD} + \lambda_v(v)\left(E(\hat{v}, BA, FD, t) - E(v, BA, FD, t)\right) \\ &+ \frac{1}{2}\sigma_v^2(v, \psi, t)E_{vv} - rE - \sigma_v(v, \psi, t)\sigma_W(K, t)\rho_{vW}E_v \Big] + E_t = 0. \end{aligned} \tag{99}$$

The supremum operator contains the two controls $\psi(t)$ and $\zeta(t)$ due to the assumption that the firm is operated as to maximize the value of the equity. The last term inside the supremum operator captures the covariance between the return on asset process and the discount factor process. In contrast to the previous models, the partial derivative of the equity value with respect to time t is not equal to zero. The firm's equity value depends explicitly on calendar time t since we assumed a finite maturity for the firm's debt. This in turn impacts the equity cash flow function and thus the value of the equity. Rearranging terms in (99) leads to (39).

Now we turn to the firm's debt. The differential for the debt value is given by

$$\begin{aligned} L^{\psi^*,\zeta^*}[D(v, BA, FD, t)] = \\ \mu_v(v, \psi^*, t)D_v + \psi^*(t)BA(t)D_{BA} + \zeta^*(t)FD(t)D_{FD} \\ + \lambda_v(v)\left(D(\hat{v}, BA, FD, t) - D(v, BA, FD, t)\right) + \frac{1}{2}\sigma_v^2(v, \psi^*, t)D_{vv}. \end{aligned} \tag{100}$$

Although the partial differential operator in (100) closely resembles the operator for the firm's equity, there exists a crucial difference. In the case of (98), the controls $\psi(t)$ and $\zeta(t)$ are chosen endogenously by the equityholders or the firm's management on their behalf, i.e. they are discretionary. However, from the point of view of the debtholders, the controls are determined exogenously by the equityholders. Thus, they are no longer discretionary but follow the optimal control laws as derived by the equityholders according to (39). Therefore, the partial differential operator for the firm's debt value is defined under these optimal controls. Using (100), the stochastic discount factor (32), the state variable processes, and the cash flow function to debt (38) in proposition 4 provides the following PDE for the debt value

$$
\begin{aligned}
& i_D FD(t) - \zeta^*(t)D + \mu_v(v,\psi^*,t)D_v + \psi^*(t)BA(t)D_{BA} \\
& + \zeta^*(t)FD(t)D_{FD} + \lambda_v(v)\left(D(\hat{v},BA,FD,t) - D(v,BA,FD,t)\right) \\
& + \frac{1}{2}\sigma_v^2(v,\psi^*,t)D_{vv} - rD - \sigma_v(v,\psi^*,t)\sigma_W(K,t)\rho_{vW}D_v + D_t = 0.
\end{aligned} \tag{101}
$$

In deriving (101), we have to use the optimal controls $\psi^*(t)$ and $\zeta^*(t)$ in the cash flow function and in all other terms since from the debtholders' point of view these are exogenous. Consequently, the supremum operator in (14) disappears since the debtholders cannot control the state variable processes. The explanations for the remaining terms correspond to those used above in the derivation of (99). Rearranging (101) gives (40). □

A.9 Proof of Proposition 9

Proof. Applying the state variable processes to proposition 4 yields

$$
\begin{aligned}
& L_i[V(R,\mu,\gamma,PPE,TL,X,t)] = \\
& \qquad \mu(t)R(t)V_R + \kappa_\mu\left(\bar{\mu} - \mu(t)\right)V_\mu + \kappa_\gamma\left(\bar{\gamma} - \gamma(t)\right)V_\gamma \\
& \qquad + \left(-DE(t) + CE(t)\right)V_{PPE} + TL_i(t)V_{TL} \\
& \qquad + \left(rX(t) + R(t) - C(t) - T_i(t) - CE(t)\right)V_X \\
& \qquad + \frac{1}{2}\sigma^2(t)R^2(t)V_{RR} + \frac{1}{2}\eta^2(t)V_{\mu\mu} + \frac{1}{2}\phi^2(t)V_{\gamma\gamma} \\
& \qquad + \sigma(t)\eta(t)\rho_{R\mu}R(t)V_{R\mu} + \sigma(t)\phi(t)\rho_{R\gamma}R(t)V_{R\gamma} + \eta(t)\phi(t)\rho_{\mu\gamma}V_{\mu\gamma} \\
& \qquad \qquad i = 1,2,
\end{aligned} \tag{102}
$$

where

$$
\begin{aligned}
& TL_1(t) = -\left(R(t) - C(t) - DE(t)\right), \ T_1(t) = 0 \qquad \text{for } R - C - DE - TL \leq 0, \\
& TL_2(t) = -TL(t), \ T_2(t) = \tau\left(R(t) - C(t) - DE(t) - TL(t)\right) \qquad \text{otherwise.}
\end{aligned}
$$

Here, we have to distinguish between two differentials as the evolution of the tax loss carry forward and the tax function differ dependent on whether the firm's pre-tax earnings $R(t) - C(t) - DE(t)$ are above or below the existing tax loss carry forward $TL(t)$. According to the assumptions, the discount factor in the economy is given by (32). Using the above differentials, the discount factor (32), the state variable processes (46), (47), (51), (55), (56), (58), and the cash flow function (59) in proposition 4 leads to the following set of PDEs for the firm value

$$\begin{aligned}
&\mu(t)R(t)V_R + \kappa_\mu \left(\bar{\mu} - \mu(t)\right) V_\mu + \kappa_\gamma \left(\bar{\gamma} - \gamma(t)\right) V_\gamma \\
&+ \left(-DE(t) + CE(t)\right) V_{PPE} + TL_i(t)V_{TL} \\
&+ \left(rX(t) + R(t) - C(t) - T_i(t) - CE(t)\right) V_X + \frac{1}{2}\,\sigma^2(t)R^2(t)V_{RR} \\
&+ \frac{1}{2}\,\eta^2(t)V_{\mu\mu} + \frac{1}{2}\,\phi^2(t)V_{\gamma\gamma} + \sigma(t)\eta(t)\rho_{R\mu}R(t)V_{R\mu} + \sigma(t)\phi(t)\rho_{R\gamma}R(t)V_{R\gamma} \\
&+ \eta(t)\phi(t)\rho_{\mu\gamma}V_{\mu\gamma} - rV - \sigma(t)\sigma_W(K,t)\rho_{RW}R(t)V_R \\
&- \eta(t)\sigma_W(K,t)\rho_{\mu W}V_\mu - \phi(t)\sigma_W(K,t)\rho_{\gamma W}V_\gamma + V_t = 0 \qquad i = 1, 2.
\end{aligned} \tag{103}$$

In deriving (103), it has to be taken into account that the supremum operator in (14) vanishes since the sets of admissible controls $\mathbb{M}$, $\mathbb{N}$, and $\mathbb{O}$ are empty. The last three terms on the left-hand side involving the partial derivatives V_R, V_μ, and V_γ capture the covariance between the state variable processes and the discount factor. They directly follow from inserting the respective process specifications into the double sum in (14). Due to the boundary condition at time T, the firm value explicitly depends on calendar time t and the partial derivative with respect to t does not disappear. Simple rearrangements of (103) then lead to (62). □

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