The Firm in a Changing Environment: About Investors' Rational Pessimism and the Consequences on Corporate Financial Decision Making *

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Abstract

The recession at the beginning of the new millennium accompanied by a series of major corporate-accounting scandals (like Enron – Arthur Andersen, WorldCom, Global Crossing, Ahold, ...) has pointed investors to the fact that (i) industry growth is apparently changing over time and (ii) the quality of signals about firms' profitability is not necessarily perfect. Despite the fact that many indicators suggest a beginning recovery of the economy in the U.S. and in Europe, investors seem to be reluctant in gaining confidence in a prosper future.

Inspired by Veronesi (1999), who gives rationale—using a general equilibrium asset pricing model—why investors underreact to good news in bad times, I develop a partial equilibrium contingent claims model of the firm in an uncertain environment. I.e., the growth rate of the cash flow generated by the firm's productive assets is modeled to change over time and the firm's equity and its debt are interpreted as claims contingent on this flow. To form their expectations about the firm's future, investors have to condition their belief about the growth perspectives on signals of certain quality. It is shown (i) how to derive the risk neutral dynamics of the Bayesian belief about growth in an partial equilibrium setting (ii) that positive risk premia imply the existence of 'rational pessimism' about growth and how this distortion in the beliefs depends on information quality, (iii) how to value equity and debt (analytically in some special cases and on a two dimensional tree in general), (iv) how investors pessimism influences equityholders decision to default, and (iv) that—as a consequence—the value of the firm's stakes and the debt capacity of the firm deteriorate if the information quality of signals about growth is low.

1 Introduction

Firms operate in an uncertain environment. Investors were pointed to this simple fact by the economic downturn at the beginning of the new millennium which demonstrated unequivocally that firms' growth rates are changing over time. Furthermore, the corporate accounting scandals which accompanied the recession (like Enron – Arthur Andersen, WorldCom, Global Crossing, Ahold, ...) exposed that the information about the profitability of firms is not necessarily as precise as pretended. Many indicators suggest that the economy in the US and partially also in Europe start to recover their strength, however, investors seem to be reluctant to regain confidence in the value of firm's equity and debt. (E.g., in the second week of October 2002, Ford Motor Company's five-year bonds quoted at a level, where they should be classified as junk, i.e., non-investment grade. Standard & Poor's as well as Moody's confirm that the firm's fundamentals do not justify such a bad valuation.[1])

[1] See The Economist Global Agenda, "Irrational gloom?", http://www.economist.com, October 11, 2002

In order to address the problem of corporate financial decision making in the context of a changing growth-environment I develop a contingent claims model of the firm where the growth rate of the firm's cash flow is uncertain. More precisely, in a partial equilibrium model I assume that there are two possible states for the firm's cash flow (high growth versus low growth) and that the cash flow changes its state due to random and nonobservable shocks. Apparently, corporate decision making like the choice of the optimal capital structure or the decision to default depend on what managers / investors belief to be the current growth rate. The paper therefore describes how to form rational beliefs about growth form the observation of historical realizations of the cash flow and from an additional signal with certain information quality and how to value the firm's equity and debt contingent on ones belief. I show that under given prices for risk, investors rational valuation of the firm's stakes is as if they would deviate from the Bayesian rule when updating their belief in the course of receiving new information. Especially if risk premia are positive then investors exhibit 'rational pessimism'.[2] Their valuation behavior is as if they where pessimistic about the firms future, i.e., they seem to expect to receive bad news with higher probability than it is objectively justified or, alternatively, it seems that they belief more in bad signals than in the good. Furthermore, the paper shows that increasing signal quality generally mitigates this effect.

These distortions of expectations about the firm's future have significant consequences on corporate decision making. I show that 'rational pessimism' leads to early bankruptcy and that this, furthermore, reduces the firm's debt capacity, i.e., the amount of debt a firm optimally issues. When studying optimal capital structure choice, I illustrate that an increase in information quality of the signal (i.e., higher credibility of data disclosed by the firm) generally increases the firm's debt capacity by reducing investor's pessimistic valuation behavior. However, the dependency of the valuation behavior is found to be not necessarily monotone in the information quality of the signal.

As initially introduced by Merton (1974), contingent claims models of the firm interpret the firm's equity and debt as claims contingent on the value of the underlying productive assets. My paper extends the existing literature by combining the option based view of the firm with the model of an uncertain growth environment as it is used in the general equilibrium models of Veronesi (1999), Veronesi (2000), and David and Veronesi (2002). Veronesi (1999) shows that investors with CARA utility over consumption overreact to bad news in good times (i.e., when growth is high), whereas they underreact to good news in bad states. My finding of distortions in investors expectations about the firms future is in accordance with these results. Veronesi (2000) studies the influence of information quality on risk premia, on return volatility, and on the relationship between conditional expected returns and conditional variance in a pure-exchange economy where identical investors have isoelastic utility. He also finds that the *value adjusted* belief distribution differs from the objective belief distribution, such that it puts more weight to the low growth states than objectively justified if they are more risk averse than the log-utility investor.

[2]The actual condition is slightly stronger, see Proposition 3.

David and Veronesi (2002) assume that investors have isoelastic utility over consumption and determine European option prices in an endowment economy. I study the firm in a partial equilibrium model (as this is done e.g., by Merton (1974), Leland (1994), and Goldstein et al. (2001)). This allows to abstract from specific utility functions of investors and does not require to explicitely restrict the economy under consideration to have a certain simple form. However, it relies on the exogenous assumption of the riskless rate of interest and the prices for risk associated with the various sources that drive the dynamics of uncertain investments.

The paper is composed as follows. Section 2 introduces the model and the assumptions made and characterizes learning about the growth rate of the firm's cash flow. In Section 3 it is presented how to value claims contingent on the firm's cash flow in a given partial equilibrium. Furthermore, the risk neutral dynamics of the cash flow and the belief are characterized. Section 4 discusses the valuation of the firm's debt and equity in the uncertain environment. Two special cases are presented in which the valuation can be done analytically and the numerical procedure for claim valuation in the general case is described. Section 5 demonstrates the effect of uncertainty and information quality on the capital structure choice of the firm. Section 6 concludes.

2 The Model

The goal of this section is to formulate a contingent claims model of the firm (in the spirit of Leland (1994)) where the firm operates in an uncertain environment. The (nontraded) underlying is the cash flow generated by the productive assets of the firm which follows an ex ante uncertain path. The value of different stakes in the firm are then interpreted as contingent claims on the cash flow.[3] In contrast to other option-based models of the firm, the cash flow process is not assumed to be a simple geometric Brownian motion. Rather than taking a constant growth rate, the growth rate is assumed to change over time (modeling the fact that the regime wherein a firm operates is not constant over time). This is done in the same way as in Veronesi (1999) and Veronesi (2000). Beginning with the formulation of the assumptions about the firm's cash flow I will then be more specific about the firms capital structure and about what happens to the firms assets in the case of bankruptcy.

Assumption 1 (cash flow) *Let x_t denote the firm's instantaneous free cash flow after corporate tax which is generated by the productive assets of the firm. Assume that its dynamics are characterized by the following stochastic differential equation*

$$\begin{aligned} \frac{dx_t}{x_t} &= \mu_t\, dt + \sigma_x\, (dW_x)_t, \\ x_0 &> 0, \end{aligned} \tag{1}$$

[3]Cash flow based contingent claims models are a convenient adaptation of the more recent firm models which take the value of the firms assets as the underlying process, this is e.g., used in the firm models of Mella-Barral and Perraudin (1997), Goldstein et al. (2001), or Dangl and Zechner (2003).

where W_x is a standard Wiener process. The volatility σ_x is a known constant, the growth rate μ_t characterizes the current growth regime. It is either low ($\mu_t = \mu_l$ in times of a downturn) or high ($\mu_t = \mu_h$ in times of prospering growth), details about the underlying process are described in Assumption 2.

Within a certain growth regime, the cash flow follows a standard geometric Brownian motion, i.e., the relative fluctuations in x_t are normally distributed. However, the mean of this distribution is not stable but changes driven by a jump process.

Assumption 2 (regime shift) *The growth rate μ_t of the cash flow process characterizes the prevailing regime (e.g., either boom or downturn) and can take one of the possible realizations μ_h and μ_l, with $\mu_h > \mu_l$.*

The transition between these states is driven by a continuous time Markov process with the transition probabilities given by

$$P(\Delta t) = \mathbb{I} + \begin{pmatrix} -\lambda_{h,l} & +\lambda_{h,l} \\ +\lambda_{l,h} & -\lambda_{l,h} \end{pmatrix} (\Delta t + o(\Delta t)), \tag{2}$$

where $\mathbb{I}$ is the identity matrix and $\lambda_{h,l}$ and $\lambda_{h,l}$ are the constant instantaneous intensity of transition from μ_h to μ_l and vice versa.

The lower the transition intensity $\lambda_{h,l}$ the higher is the persistence of the high growth state. The magnitude of $\lambda_{l,h}$ determines the persistence of the low growth state. Not only the growth environment the firm operates in is uncertain, the current state μ_t is hidden.

Assumption 3 (observability) *It is assumed that investors/managers are able to observe the cash flow x_t. The growth regime (i.e., the growth rate μ_t) is assumed to be not observable.*

The firm's actual problem is that when observing a certain increase / decrease in the cash flow, it is not immediately clear which proportion of this observed change comes from the currently prevailing growth rate μ_t (see the first term of the right hand side of Equation (1)) and which proportion comes from the innovation term $(W_x)_t$ (see the second term of the right hand side of Equation (1)). However, the distinction between these possible sources is of core relevance when forming expectations about the firm's future, i.e., when pricing different stakes of the firm. If observing the gradual realization of the cash flow process is the only source of information about growth, then apparently the quality of information about the current growth rate is the more accurate the lower the instantaneous variance of the innovation term σ_x. In which case information quality and riskiness of the firm are coupled. However, the cash flow itself is not necessarily the only source of information about the growth rate, thus, assuming an additional source of information allows to to distinguish between the effects of riskiness of the cash flow and the effects of the quality of information about the growth scenario. One can think of this signal as a report on the profitability of firms which operate in the same industry and, thus, share the same growth framework, or a sector index carrying information about growth perspectives. Alternatively, the signal may be extracted from information other than cash flow data which are disclosed by the firm.

Assumption 4 (informative signal) *The signal s_t is an additional source of information about growth. It follows the process*

$$\begin{aligned} \frac{ds_t}{s_t} &= \mu_t\, dt + \sigma_s\, (dW_s)_t, \\ s_0 &> 0, \end{aligned} \tag{3}$$

where W_s is a Wiener process correlated to W_x, such that $(dW_x)_t\,(dW_s)_t = \rho dt$, with $-1 < \rho < 1$. To be consistent with Assumption 3 (i.e., μ_t is a hidden state variable), $(W_s)_t$ is not observable.

The signal s is noisy and σ_s, the parameter that determines the instantaneous variance of the signal, also characterizes the information content inherent in s and, thus, it defines the extent to which managers/investors give credence to the data disclosed by the firms. This allows me to discuss the issue of general credibility of accounting data by varying σ_s without changing the riskiness of the cash flow.

Definition 1 (information set) *Let $\mathcal{F}_t$ be the canonical nondecreasing filtration jointly created by the cash flow process x_t and the signal s_t.*

In other words, the assumption about observability (Assumption 3) states that μ_t, dW_x, and dW_s are not measurable with respect to the information set $\mathcal{F}_t$.

After defining the framework in which the firm operates there are some assumptions regarding the capital structure of the firm under consideration.

Assumption 5 (capital structure) *The firm has a simple capital structure such that—beside equity—there is only one perpetual bond which requires a continuous coupon flow c in order to satisfy bondholders.*

From the various reasons why a firm might find it optimal to issues debt, I assume the existence of tax shields as the driving force.

Assumption 6 (tax advantage of debt) *Interest payments are assumed to be tax deductible.*

Since the cash flow generated by the firm is after tax, the tax advantage of debt reduces the effective coupon flow from c to $(1-\tau)c$, where τ denotes the corporate tax rate. This formulation seems to cause rising firm values when corporate tax increases, however, it implies only that the value of the levered firm increases relative to its unlevered pendant, which is obvious due to an increase in the value of the tax shield.[4]

The issue of debt capital implies that the firm bears the risk of bankruptcy, or alternatively that there is an incentive to renegotiate the debt contract in the case the firm value deteriorates.[5] Furthermore,

[4]This construction, which is also used in Fischer et al. (1989), Leland (1994) and others, is sometimes target of critiques when the effect of rising taxes is misinterpreted as discussed. Starting from the unlevered cash flow as it is done in Goldstein et al. (2001) can prevent the misinterpretation, however, it requires the modeling of the tax system in greater detail.

[5]This is considered in the models of Mella-Barral and Perraudin (1997), Mella-Barral (1999), and Christensen, Flor, Lando, and Miltersen (2000).

equityholders might want to issue new debt in the case the firm value evolves favorably.[6] However, in order not to overload the paper, formal bankruptcy is assumed to be the only way to cancel the contractual obligation of equityholders to debtholders. Equityholders do not face liquidity constraints, so bankruptcy is triggered when they decide to refrain from paying the contracted coupon.

Assumption 7 (bankruptcy) *In the case of bankruptcy the productive assets are handed over to debtholders. Debtholders sell the assets for the market value of the unlevered cash flow. They bear the costs of the bankruptcy procedure which are assumed to be a fraction* ϕ *of the proceeds.*

2.1 Learning About Expected Growth

When valuing the cash flow of the firm's productive assets or when valuing claims contingent on these cash flow, one has to form rational beliefs about the growth regime (either high growth, $\mu_t = \mu_h$, or low growth, $\mu_t = \mu_l$) in which the firm is currently operating. From observing the cash flow and the signal, one can apply filter techniques based on Bayes' theorem in order extract information about the prevailing environment.

Definition 2 (belief) *Let* π_t *denote the probability that* $\mu_t = \mu_h$ *conditional on* $\mathcal{F}_t$*, i.e., the rational belief about the cash flow growth rate* μ_t *at time* t *conditional on the observation of the cash flow* x *and the signal* s *over the time interval* $[0,t)$ *(and for a given the prior* π_0*).*

When having a certain belief π_t, the expected instantaneous growth rate of the firm's cash flow, $\frac{1}{dt}E(\frac{dx}{x})$, (as well as the growth rate of the signal) is given by $\pi_t\mu_h + (1-\pi_t)\mu_l$. Deviations of the observed instantaneous growth from this expectation gives rise to updating the belief. Before characterizing the dynamics of the Bayesian belief about the environment, it is convenient to make use of simple noise reduction techniques in order to reduce the dimension of the inference problem. For this sake define **1** as vector of ones and let the variance-covariance matrix V be

$$V = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_s \\ \rho\sigma_x\sigma_s & \sigma_s^2 \end{pmatrix}. \tag{4}$$

Corollary 1 (noise reduction) *Let* y_t *denote the compound process composed from a weighted sum of the cash flow process* x_t *and the signal* s_t

$$y_t = w' \begin{pmatrix} \ln(x_t) + \frac{1}{2}\sigma_x^2 t \\ \ln(s_t) + \frac{1}{2}\sigma_s^2 t \end{pmatrix}, \tag{5}$$

[6] Capital structure dynamics of this kind are discussed in Fischer et al. (1989), Goldstein et al. (2001), and Dangl and Zechner (2003).

where the weights w are given by

$$w = \begin{pmatrix} w_x \\ w_s \end{pmatrix} = \frac{1}{\mathbf{1}'V^{-1}\mathbf{1}} V^{-1}\mathbf{1} = \begin{pmatrix} \frac{\sigma_s^2 - \rho\sigma_x\sigma_s}{\sigma_x^2 + \sigma_s^2 - 2\sigma_x\sigma_2} \\ \\ \frac{\sigma_x^2 - \rho\sigma_x\sigma_s}{\sigma_x^2 + \sigma_s^2 - 2\sigma_x\sigma_2} \end{pmatrix}. \tag{6}$$

Then y_t follows a Brownian motion which drift μ_t that exhibits the minimum attainable volatility (or in other words the maximum attainable signal to noise ratio). The dynamics of this minimum attainable volatility compound process follows

$$\begin{aligned} dy_t &= w_x \frac{dx}{x} + w_s \frac{ds}{s} \\ &= \mu_t \, dt + \sigma_y \, (dW_y)_t, \end{aligned} \tag{7}$$

$$y_0 \qquad \textit{given by Equation (5)}$$

where σ_y^2 is the minimum attainable variance and W_y is a standard Wiener process given by

$$\sigma_y = \frac{1}{\sqrt{\mathbf{1}'V^{-1}\mathbf{1}}} = \sqrt{\frac{(1-\rho^2)\sigma_x^2\sigma_s^2}{\sigma_x^2 + \sigma_s^2 - 2\rho\sigma_x\sigma_s}}, \tag{8}$$

$$(dW_y)_t = \sigma_y \mathbf{1}'V^{-1} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_s \end{pmatrix} \begin{pmatrix} dW_x \\ dW_s \end{pmatrix}_t = \frac{1}{\sigma_y} \left[w_x\sigma_x(dW_x)_t + w_s\sigma_s(dW_s)_t \right] \tag{9}$$

PROOF: The proof is provided in the Appendix.

We will see that this *minimum attainable volatility* compound process y_t contains all the information about the growth rate μ_t that is jointly contained in both, the cash flow x_t and the signal s_t. Consequently, the dynamics of y_t determines the dynamics of the Bayesian belief π_t, whereby the instantaneous variance of y_t characterizes the quality of available information. When calculating the minimum variance portfolio composed of a set of available assets, the vector of weights is exactly the same as w in Corollary 1. With the help of y_t it is possible to reduce the inference problem to a one dimensional problem.[7]

Corollary 2 (learning about growth) *It is sufficient to condition learning on the observed realization of of the minimum volatility compound process y_t.*

The dynamics of the Bayesian belief about the growth scenario π_t then follows from the theorem on

[7] This remaines true even if there are several signals that contain information about the growth rate. The expressions in matrix form in Corollary 1 allow for an easy extension.

nonlinear filtering in Liptser and Shiryaev (2000), Chapter 9,

$$\begin{aligned} d\pi_t &= [-\pi\lambda_{hl} + (1-\pi)\lambda_{lh}]dt + (\mu_h - \mu_l)\pi_t(1-\pi_t)\frac{1}{\sigma_y}(dW_y^{\mathcal{F}})_t, \\ \pi_0 &= \pi_{a\ priori}, \end{aligned} \tag{10}$$

$(W_y^{\mathcal{F}})_t$ is a Wiener process with respect to $\mathcal{F}_t$ given by

$$\begin{aligned} (dW_y^{\mathcal{F}})_t &= \frac{1}{\sigma_y}\left[dy - E_t(\mu_t|\mathcal{F}_t)dt\right], \\ (W_y^{\mathcal{F}})_0 &= 0. \end{aligned}$$

PROOF: The proof is provided in the Appendix.

The dynamics of the belief π_t are characterized by a mean-reverting diffusion which is bound between 0 (corresponding to the proposition $\mu = \mu_l$, almost sure) and 1 ($\mu = \mu_h$, almost sure). The first therm in Equation (10) describes the dilution of information over time due to fact that the growth regime tends to change (see Assumption 2). This causes the belief to revert towards the unconditional belief

$$[-\pi\lambda_{hl} + (1-\pi)\lambda_{lh}] = 0 \quad \Leftrightarrow \quad \pi_{\text{unconditional}} = \frac{\lambda_{lh}}{\lambda_{hl} + \lambda_{lh}}. \tag{11}$$

The diffusion term in Equation (10) characterizes learning from observations of the informative signal y_t. The increment of the Wiener process $(W_y^{\mathcal{F}})_t$ represents the deviation of y_t from its expected growth-path. Whenever y_t grows at a higher rate than expected ($(dW_y^{\mathcal{F}})_t > 0$), the second term in Equation (10) yields a positive contribution to $d\pi_t$, i.e., a contribution that strengthens the belief in high growth. On the contrary, whenever y_t grows at a lower rate than expected ($(dW_y^{\mathcal{F}})_t < 0$), the second term in Equation (10) yields a negative contribution to $d\pi_t$ pushing the belief towards the low growth scenario. The extend to which the belief reacts to new information $(dW_y^{\mathcal{F}})_t$ is given by the term $(\mu_h - \mu_l)\pi_t(1-\pi_t)\frac{1}{\sigma_y}$. The learning effect is the more pronounced (i) the greater the difference $(\mu_h - \mu_l)$ between the two possible scenarios, (ii) the lower the information content of the current belief (i.e., the larger $\pi_t(1-\pi_t)$), and (iii) the lower the volatility of the signal y_t (from Corollary 1 we know that y_t is the minimum variance compound process). If one abstains from learning (which corresponds to dropping the diffusion term in (10)) then the information is diluted gradually and converges to the unconditional belief about growth.

The fact that $(W_y^{\mathcal{F}})_t$ is a Wiener process with respect to the filtration $\mathcal{F}_t$ means that all information about the growth rate available at time t is already contained in the current belief π_t, or in other words, if there were systematic information on the gradual unfolding of information in the future, this systematic information could be used to immediately update the current belief. Therefore, the belief π follows a $\mathcal{F}_t$-Markov process. Alternatively, one can view $(W_y^{\mathcal{F}})_t$ as the *perceived innovation / noise* of the informative signal, i.e., the disturbance that drives the deviation from the expected drift.

Since pricing of claims contingent on the firms cash flow will be done with respect to the information investors can access (i.e., with respect to $\mathcal{F}_t$), it is convenient to characterize the dynamics of the cash flow and the signal with respect to the time t expectation about growth.

Definition 3 *Let $W_x^{\mathcal{F}}$ and $W_s^{\mathcal{F}}$ denote the $\mathcal{F}_t$-Wiener processes defined by*

$$(dW_x^{\mathcal{F}})_t = \frac{1}{\sigma_x}\left[\frac{dx}{x} - E_t(\mu_t|\mathcal{F}_t)dt\right], \quad (dW_s^{\mathcal{F}})_t = \frac{1}{\sigma_x}\left[\frac{ds}{s} - E_t(\mu_t|\mathcal{F}_t)dt\right]. \tag{12}$$

Proposition 1 (dynamics with respect to $\mathcal{F}_t$) *The joint dynamics of the cash flow x, the signal s, and the minimum volatility compound process y with respect to the time t expectation about growth are*

$$\begin{aligned} \begin{pmatrix} dx/x \\ ds/s \\ dy \end{pmatrix}_t &= [\pi_t\mu_h + (1-\pi_t)\mu_l]\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dt + \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_s & 0 \\ 0 & 0 & \sigma_y \end{pmatrix}\begin{pmatrix} dW_x^{\mathcal{F}} \\ dW_s^{\mathcal{F}} \\ dW_y^{\mathcal{F}} \end{pmatrix}_t, \\ \begin{pmatrix} x \\ s \\ y \end{pmatrix}_0 &= \begin{pmatrix} x_0 \\ s_0 \\ y_0 \end{pmatrix}, \end{aligned} \tag{13}$$

where π_t is updated as stated in Corollary 2. The correlation between these processes is such that

$$dW_x^{\mathcal{F}} dW_y^{\mathcal{F}} = \frac{\sigma_y}{\sigma_x}dt, \quad dW_s^{\mathcal{F}} dW_y^{\mathcal{F}} = \frac{\sigma_y}{\sigma_s}dt. \tag{14}$$

PROOF: The proof is provided in the Appendix.

3 Claim Dynamics and Claim Valuation

From the previous section we know that the Bayesian belief π_t about growth follows a $\mathcal{F}$-Markov process. Thus, when starting from a certain prior, all information about growth that can be extracted from the observed path of the minimum variance compound process y is contained in the current belief π_t. Therefore any claim contingent on the cash flow of the firm can be interpreted as a function of the current level of the cash flow, x_t, and the current belief, π_t. Since the firms equity and debt are modeled as claims contingent on the firm's cash flow, the following equations can directly be applied to the valuation of the firm's stakes.

Proposition 2 (claim value dynamics) *Let F denote the value of a claim contingent on the cash flow of the firm's productive assets x_t. Since the valuation depends on the prevailing growth rate regime*

($\mu_t \in \{\mu_h, \mu_l\}$), F is a function of x and π, and its dynamics are given by

$$\begin{aligned} \frac{dF_t}{F} &= \alpha_F\, dt + \sigma_{Fx}\,(dW_x^{\mathcal{F}})_t + \sigma_{F\pi}(dW_y^{\mathcal{F}})_t, \\ F_0(x,\pi) &= \Phi(x,\pi). \end{aligned}$$

where

$$\begin{aligned} (\alpha_F)_t &= \frac{1}{F}\left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}x(\pi\mu_h + (1-\pi)\mu_l) \right. \\ &\quad + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}x^2\sigma_x^2 + \frac{\partial F}{\partial \pi}(-\pi\lambda_{hl} + (1-\pi)\lambda_{lh}) \\ &\quad \left. + \frac{1}{2}\frac{\partial^2 F}{\partial \pi^2}\pi^2(1-pi)^2(\mu_h-\mu_l)^2\frac{1}{\sigma_y^2} + \frac{\partial^2 F}{\partial x \partial \pi}x\pi(1-\pi)(\mu_h-\mu_l)\right], \\ (\sigma_{Fx})_T &= \frac{1}{F}\left[\frac{\partial F}{\partial x}x\sigma_x\right], \\ (\sigma_{F\pi})_t &= \frac{1}{F}\left[\frac{\partial F}{\partial \pi}\pi(1-\pi)(\mu_h-\mu_l)\frac{1}{\sigma_y}\right]. \end{aligned}$$

PROOF: The proof is provided in the Appendix.

These are the claim dynamics of a general claim contingent on the firm's cash flow under the objective probability measure. It can be seen that the claim value and the underlying cash flow are perfectly correlated only when there is precise knowledge about the growth state, i.e., if $\pi_t = 0$ or $\pi_t = 1$. Otherwise, the claim value is also driven by the realization of the signal. When the knowledge about the growth rate is imprecise, the signal influences the belief fluctuation and, thus, increases the volatility. Furthermore, this dependency of the claim contract on the signal reduces the correlation of F_t to x_t and adds a further innovation source to the claim dynamics. This fact has serious consequences on the valuation of claims. The first is technical and regards the completion of the market, the second concerns the impact of volatility coming from the Bayesian updating process on the pricing of a contract (see the discussion of Proposition 4 for more detail on this topic).

Since investors are usually not risk neutral, pricing cannot be done by computing expectations under the objective probability measure. The dynamics of a claim value is driven by two sources of innovation, dW_x and dW_y, therefore, it requires in an arbitrage free market two traded contracts in addition to an instantaneously riskless investment opportunity to reveal the market prices for risk associated with $W_x^{\mathcal{F}}$ and $W_y^{\mathcal{F}}$ respectively.[8] Then the market is complete such that any further claim contingent on x_t can be valued in a preference free manner.

Assumption 8 (market conditions) *The market is assumed to be arbitrage free. Let r_t denotes the instantaneous riskless rate of interest. The processes $(\kappa_x)_t$ and $(\kappa_s)_t$ denote the market prices for risk*

[8]These contracts have to be claims contingent on the firm's cash flow like equity, debt or a claim to a share of the cash flow of the firms unlevered assets or a contract written on an underlying perfectly correlated to it. There are regularity conditions to be met by these two traded contracts such that the respective market prices for risk can be uniquely determined. These conditions are assumed to be satisfied.

associated with the innovation of of the cash flow, $W_x^{\mathcal{F}}$, and of the signal, $W_s^{\mathcal{F}}$, respectively. And it is assumed that $(\kappa_x)_t \geq 0$. Since the cash flow of the productive assets is assumed to be positive, equilibrium requires that $\mu_h - (\kappa_x)_t \sigma_x < r_t$.

Please noted that $(\kappa_x)_t$ and $(\kappa_s)_t$ measure the price for risk with respect to the time-t expectation, i.e., the compensation investors require for the dispersion of the claim value around the expected value gain.

Proposition 3 (risk neutral valuation) *Suppose a claim provides a cash flow of $kx_t + d$ as long as it is alive, then the market conditions described in Assumption 8 require that the value function F has to satisfy the SDE*

$$rF = kx_t + d + F \cdot (\alpha_F)_t - F \begin{pmatrix} \kappa_x \\ \kappa_\pi \end{pmatrix}_t' \begin{pmatrix} \sigma_{Fx} \\ \sigma_{F\pi} \end{pmatrix}_t, \tag{15}$$

where $(\kappa_\pi)_t$ is the market price of risk associated with fluctuations in the Bayesian belief given by

$$\begin{aligned} (\kappa_\pi)_t &= \sigma_y \mathbf{1}' V^{-1} \begin{pmatrix} (\kappa_x)_t \sigma_x \\ (\kappa_s)_t \sigma_s \end{pmatrix} \\ &= \frac{1}{\sigma_y} [(\kappa_x)_t w_x \sigma_x + (\kappa_s)_t w_s \sigma_s], \end{aligned} \tag{16}$$

and the terms $(\alpha_F)_t$, $(\sigma_{Fx})_t$, and $(\sigma_{F\pi})_t$ are those stated in Proposition 2.
Investors are risk neutral with respect to belief fluctuations if it is true that

$$\sigma_s[\kappa_s \rho - \kappa_x] = \sigma_x[\kappa_s - \rho\kappa_x] \tag{17}$$

and, furthermore, κ_π is nonnegative, independent of the volatilities σ_x and σ_s if

$$\begin{aligned} &\kappa_s - \rho\kappa_x \geq 0 \quad \textit{and} \quad \kappa_x - \rho\kappa_s \geq 0 \quad \textit{if } \rho \geq 0 \\ &\kappa_s - \rho\kappa_x \geq 0 \qquad\qquad\qquad\qquad\quad \textit{if } \rho < 0 \end{aligned} \tag{18}$$

PROOF: The proof is provided in the Appendix.

The proposition states that the claim value F has to satisfy the partial differential equation (15), such that F is—in general– a function of the cash flow x, the belief about growth π and of time t. However, if the firm's cash flow is modeled to be not explicitely time dependent, Equation (15) is a second order partial differential equation in x and π.

From Proposition 2 we know that the volatility of a claim changes over time, depending on the precision of the knowledge about the growth rate. Moreover, the correlation properties of the claim value change when the belief π changes. Whereas the claim value is perfectly correlated to the cash flow if $\pi = 0$ or $\pi = 1$ this is not true when the belief is not exact. Thus, the price for risk inherent in a contract is only equal to κ_x if one knows the growth rate exactly. Otherwise, the Bayesian updating process adds further uncertainty and changes the price for risk associated with the volatility of the contract. The price for the excess volatility coming from belief fluctuations depends, of course, on the price for risk associated with the signal κ_s, but also on the weights w_x and w_s one optimally applies in the creation of the minimum volatility compound process y on which leaning is conditioned (see Corollary 1). That means, the price for his excess volatility depends on the correlation between the cash flow and the signal and on the volatility of both. Ceteris paribus, κ_π is not necessarily monotonous in the information quality σ_s of the signal. In the case the signal carries no information (e.g., if $\rho = 1$ and $\sigma_s = \sigma_x$), then $\kappa_\pi = \kappa_x > 0$.

Equation (18) states the conditions under which the price for risk in the belief evolution is nonnegative. Note that when decomposing dW_s into a component that is perfectly correlated with dW_x and a component that is orthogonal to it then $\kappa_{x_\perp} = (\kappa_s - \rho\kappa_x)/\sqrt{1-\rho^2}$ is the price of the risk associated with the orthogonal component. Analogously, $\kappa_{s_\perp} = (\kappa_x - \rho\kappa_s)/\sqrt{1-\rho^2}$. is the price for risk of the component in dW_x that is orthogonal to dW_s. Therefore, for nonnegative correlation between cash flow and signal, $\rho \geq 0$, condition (18) says that if the prices for risk related to these orthogonal components are both nonnegative, then κ_π is nonnegative.

For negative correlation $\kappa_{x_\perp} \geq 0$ is sufficient to ensure $\kappa_\pi \geq 0$ (since this implies $\kappa_{s_\perp} \geq 0$). Therefore, if risk premia in the partial equilibrium under consideration are positive in the sense of condition (18), then the risk premium associated with belief fluctuations is positive.[9]

If condition (18) is violated, this means that some fraction of the risk in x or s has negative price of risk. Then it is possible—under a specific constellation of the weights w_x and w_s in the compound process y—that the negative price of this risk component outweighs the positive price of the other. However, the sign of κ_π depends on the informativeness of the signal σ_s. The price for risk κ_π at the extrema is also determined by the prices for the orthogonal components

$$\begin{aligned} \lim_{\sigma_s \to 0} \kappa_\pi &= \frac{\kappa_s - \rho\kappa_x}{\sqrt{1-\rho^2}} = \kappa_{x_\perp} \\ \lim_{\sigma_s \to \infty} \kappa_\pi &= \frac{\kappa_x - \rho\kappa_s}{\sqrt{1-\rho^2}} = \kappa_{s_\perp} \end{aligned} \tag{19}$$

The dynamics of the cash flow x and the belief π can be transformed into risk neutral dynamics under which pricing can be performed with respect to expected values. These are stated in the following proposition

[9] However, this condition does not require $\kappa_s \geq 0$.

Proposition 4 (risk neutral dynamics) *It exists a unique martingale measure such that the risk neutral dynamics of the cash flow x_t and of the belief π_t under this measure are given by*

$$\begin{aligned} \frac{dx_t}{x_t} &= [\pi_t \mu_h + (1-\pi_t)\mu_l - (\kappa_1)_t \sigma_x]\, dt + \sigma_x\, (dW_x^{\mathcal{F}})_t, \\ x_0 &> 0, \end{aligned} \tag{20}$$

$$\begin{aligned} d\pi_t &= \left(-\pi\lambda_{hl} + (1-\pi)\lambda_{lh} - (\kappa_\pi)_t \frac{\pi_t(1-\pi_t)(\mu_h-\mu_l)}{\sigma_y}\right) dt \\ &\quad + (\mu_h-\mu_l)\pi_t(1-\pi_t)\frac{1}{\sigma_y}(dW_y^{\mathcal{F}})_t, \\ \pi_0 &= \pi_{a\ priori}. \end{aligned} \tag{21}$$

PROOF: The proof is provided in the Appendix.

The measure transformation for both, the cash flow process and the belief process, is done by adaptation of the drift rate, i.e., by subtraction of the market price for risk times the instantaneous standard deviation of the process. While the effect of the measure transformation on the cash flow process is well known, the effect on Bayesian learning is less obvious. If κ_π, the price for information risk, is nonzero, then Bayesian learning under the risk neutral measure exhibits a distortion compared to Bayesian learning under the objective probability measure. I.e., investors value claims as if they would deviate from the Bayesian rule when updating their belief or, alternatively, if they would expect good news about cash flow growth with unjustified high or low probability, and thus, investors seem to act irrationally.

The direction of this distortion depends on the sign of κ_π. If risk premia are positive in the sense of condition (18), investors valuation is as if they were reluctant to belief in the high growth scenario and tend to shift their belief towards the low growth scenario. Thus, their valuation is *rationally pessimistic* independent of the signal quality, i.e., investors rationally act as if bad news arrive with higher probability than it is suggested by objective statistics. Or alternatively, they seem to belief more in the bad signals than in good signals. This result can be interpreted as the *partial equilibrium version* of the finding Veronesi (1999) presents in a general equilibrium consumption based asset pricing model under the assumption of a CARA utility function. With his model he gives rationale why investors show overreaction to bad news in good times while they do not react in the same intensity to good news in bad times. It is also consistent with the findings of Veronesi (2000) that—again in a consumption based general equilibrium model—the *value adjusted* belief distribution differs from the objective. Assuming isoelastic utility his model predicts that investors put more weight to the low growth states if they are more risk averse than the log-utility investor.

In the partial equilibrium model the price for risk associated with belief fluctuations is—due to the interference with the exogenous part of the economy—is not necessarily positive. If (18) is not satisfied, investors may show *rational optimism*, i.e., they value claims as if they would belief more in good signals than in the bad. However, this effect can only occur if a fraction of the risk in the cash flow or the signal has a negative price of risk. The instantaneous effect of this distortion is the more pronounced the higher the investors uncertainty about the current growth scenario is, i.e., the higher $\pi_t(1-\pi_t)$ (see Equation 21).

A second, more technical, consequence of the distortion on belief dynamics is that aversion with respect to information risk qualitatively changes the model solution compared to the case of risk neutrality concerning uncertainty about growth (i.e., with $\kappa_\pi = 0$). In traditional contingent claims models of the firm which do not address the issue of learning, nonzero prices for risk do not necessarily change the form of the model solution. The volatility of the cash flow, σ_x, is assumed to be a constant, therefore, if the market price for risk is constant, the measure transformation of the cash flow process is simply a static correction of the objective drift. Thus, model solutions with constant prices for risk can simply be obtained from the solutions under risk neutrality by adaptation of the drift rate. This is not true for the firm model in an uncertain environment. The instantaneous standard deviation of the belief process, $\pi_t(1-\pi_t)(\mu_h-\mu_l)/\sigma_y$ fluctuates over time, such that the measure transformation is not a static adaptation, even in the case of a constant price for risk, and therefore qualitatively changes the model solution, as we will see in the remainder of the paper.

After deriving the dynamics of the belief the question about the long run distribution of π can be addressed. How does information quality and prices for risk influence the risk neutral probability that the belief is in a certain range? This is a question about the stationary probability density of the belief π under the risk neutral probability measure.[10]

Proposition 5 (stationary belief density) *Let $\varphi(p)$ denote the risk neural stationary probability density of the belief π_t, i.e., $\varphi(p)$ characterizes the fraction of time investors' belief under the risk neutral probability measure is in the interval $[p, p+dp]$ (in the long run). Assume that prices for risk are constant, then the stationary density $\varphi(p)$ is given by*

$$\varphi(p) = C\frac{e^{-k_1(p)}\left(\frac{1-p}{p}\right)^{k_2}}{(1-p)^2p^2} \tag{22}$$

where C is an integration constant which ensures that the probability density integrates to unity and the exponents $k_1(p)$ and k_2 are given by

$$k_1(p) = \frac{2\left(\frac{\lambda_{hl}}{1-p}+\frac{\lambda_{lh}}{p}\right)\sigma_y^2}{(\mu_h-\mu_l)^2}, \tag{23}$$

$$k_2 = \frac{2\left[\lambda_{hl}-\lambda_{lh}+\kappa_\pi\frac{(\mu_h-\mu_l)}{\sigma_y}\right]}{(\mu_h-\mu_l)^2} \tag{24}$$

PROOF: The proof is provided in the Appendix.

[10] Investors, of course, use the objective Bayesian rule when updating their belief about growth, however, when valuing claims, they act as if they would update their belief in an irrational manner, as described in Assumption 4.

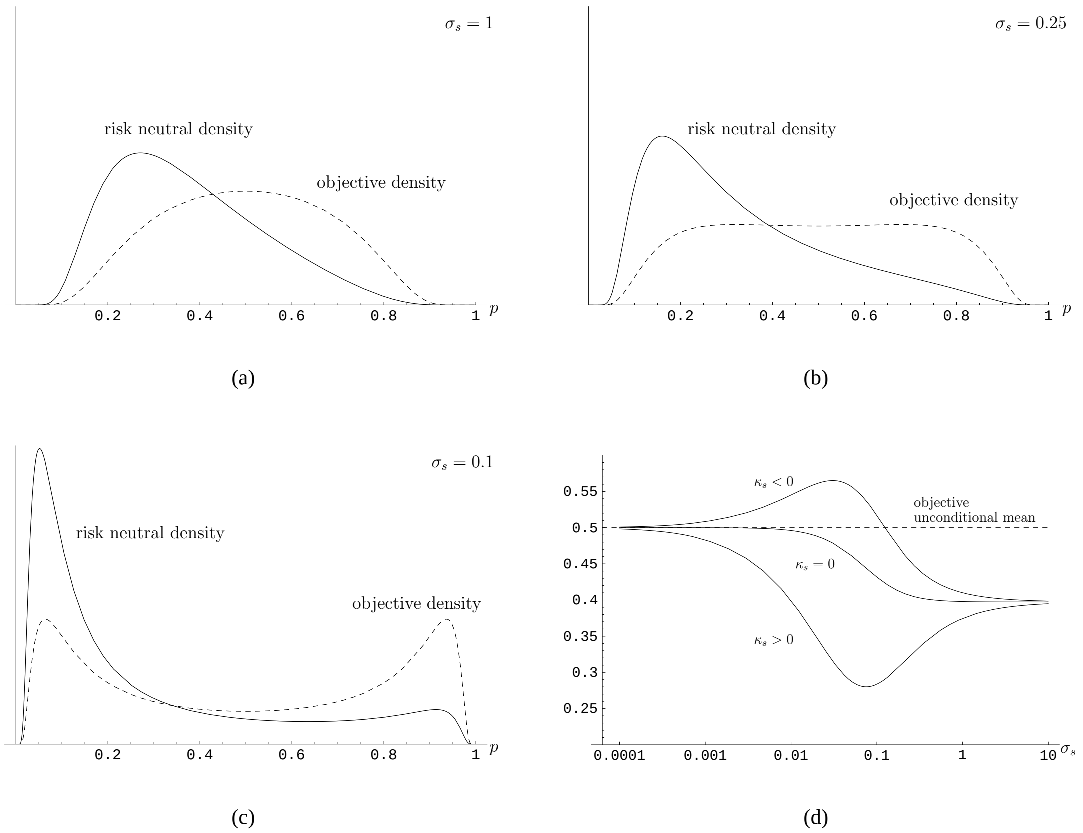

Figure 1: For given cash flow volatility σ_x and prices of risk κ_x, κ_s Panels (a) to (c) compare the stationary belief density under the risk neutral measure if risk premia are positive in the sense of condition (18), plotted as solid curves, with the stationary belief density under the objective measure (dashed curves) for different levels of signal information quality σ_s in a symmetric framework (i.e., for $\lambda_{hl} = \lambda_{lh}$). While increasing information quality increases the probability weights at states that represent precise estimates (i.e., near $p = 0$ or $p = 1$), Bayesian learning under the risk neutral measure exhibits *rational pessimism* if $\kappa_\pi \geq 0$ such that it puts higher weights to the low growth scenario than objective learning suggests. Panel (d) shows the unconditional mean of the stationary density as a measure of the deviation of the belief updating process under the risk neutral measure from the objective Bayesian rule. The three curves are computed for uncorrelated signals. When risk premia are positive in the sense of Condition (18), investors show overall pessimism (since $\rho = 0$ this is the case for the two curves with $\kappa_s \leq 0$). If condition (18) is not satisfied, investors behavior can either be pessimistic or optimistic (see curve for $\kappa_s < 0$). High information quality gives no space for these distortions on learning.

For given prices of risk κ_x and κ_s Figures 1(a) to 1(c) illustrate investors' *rational pessimism* when risk premia are positive in the sense of condition (18). The chosen framework is symmetric, i.e., $\lambda_{hl} = \lambda_{lh}$, therefore the objective belief densities are symmetric around 0.5, independent of the information quality. When information quality is low, i.e., high σ_s, the density is concentrated in the center of the interval $[0,1]$ which means that most of the time investors do not have a precise estimate of the true growth rate. High information quality shifts weights form the center to the boundary, thus, there is a higher probability that investors beliefs have high precision. Under risk neutral learning which is relevant for

claim	notation	flow to the claimholder
unlevered assets	$X(x,\pi)$	x_t
the firm's equity	$E(x,\pi)$	$x_t-(1-\tau)c$
the firm's debt	$D(x,\pi)$	c

Table 1: The different claims under consideration in the firm model, notation and flow they provide to the claimholder.

valuation, weights are shifted towards the low growth scenario. Thus, investors seem to be pessimistic about the growth perspectives of the firm. Panel 1(d) plots the mean of the stationary densities as a measure of the deviation of risk neutral learning from the Bayesian rule for different information quality. The mean of the objective belief density is stable at 1/2 (because of the symmetric setting $\lambda_{hl}=\lambda_{lh}$). The plotted curves are computed under the assumption that x and s are uncorrelated, thus, κ_π is nonnegative, i.e., (18) is satisfied, whenever $\kappa_s\geq 0$. In which case the mean of the risk neutral belief is well below the objective value (see the two curves with $\kappa_s>0$). If (18) is violated (which is the case for $\kappa_s<0$), the sign of κ_π depends on the information quality of the signal. However, if the informativeness of the signal is very high, then there is no room for these distortions in the investors behavior, independent of the correlation and risk premia. This is true because the adaptation of the drift of the risk neutral belief dynamics associated with κ_π (see Proposition 4) vanishes for $\pi\rightarrow 0$ and $\pi\rightarrow 1$. Since high information quality concentrates the belief either in the neighborhood of 0 or in the neighborhood of 1, investors' risk neutral belief dynamics do not deviate from the objective dynamics when information quality is good.

To analyze the effect of the distortions in the investors belief dynamics on the valuation of the firm's stakes and the consequences on capital structure choice and on the riskiness of debt contracts is the task of the following sections.

4 The Firm in an Uncertain Environment

After the technical requirements for the model analysis are prepared, this section focuses on the valuation of the firm's stakes. As already mentioned, equity and debt of the firm are interpreted as claims contingent on the level of the cash flow, x_t, and the belief about the growth scenario, π_t, which is extracted from x_t and an informative signal s_t. The valuation is done in a partial equilibrium, i.e., the prices for risk associated with fluctuations in the cash flow and the signal are assumed as given (by the market, see Assumption 8). These claims are special cases of the general claim that is used to derive the arbitrage free valuation equation in Proposition 3. The differences in the flow these claims provide to the claimholders is stated in Table 1. There is no explicite dependency on time t, consequently, the value functions of the firm's stakes do not explicitely depend on time, i.e., their partial derivative with respect to t vanishes.

Two special cases allow for an analytical solution of the valuation equation. These solutions can be used to serve as a benchmark for the general solution that is done numerically on a two-dimensional binomial tree. These two cases are discussed in the following subsection.

4.1 Perfect Information

In this subsection I assume that the informativeness of the signal s_t is perfect. This special case constitutes the benchmark for $\sigma_s \to 0$ which means that the growth rate is actually observable. Thus, the belief about the growth rate can take only the two extreme values $\pi = 0$ (in times of low growth) and $\pi = 1$ (in times of high growth). In the previous section I demonstrated that the mean of the risk neutral belief distribution converges to the objective mean. To be consistent with this fact, I assume in this subsection that investors are risk neutral with respect to the jumps in the growth rate. Furthermore, I assume that the riskless interest rate r is constant over time and that the prices for risk κ_x and κ_s are constant and not state dependent.

Proposition 6 (claim valuation under perfect information) *Consider a claim that provides a flow of $kx_t + d$ to the claimholder. Let $F_0(x)$ and $F_1(x)$ denote the value function of equity conditional on $\pi = 0$ and $\pi = 1$ respectively. Then these two functions must satisfy the following system of ordinary differential equations*

$$\begin{aligned}(r+\lambda_{hl})F_1(x) - \lambda_{hl}F_0(x) &= \tfrac{1}{2}\sigma_x^2 x^2 \frac{\partial^2}{\partial x^2}F_1(x) + (\mu_h - \kappa_x\sigma_x)x\frac{\partial}{\partial x}F_1(x) + kx_t + d \\ (r+\lambda_{lh})F_0(x) - \lambda_{lh}F_1(x) &= \tfrac{1}{2}\sigma_x^2 x^2 \frac{\partial^2}{\partial x^2}F_0(x) + (\mu_h - \kappa_x\sigma_x)x\frac{\partial}{\partial x}F_0(x) + kx_t + d\end{aligned} \quad (25)$$

PROOF: The proof is provided in the Appendix.

Consistent with existing contingent claims models of the firm (see e.g., Goldstein et al. (2001)) I interpret the firm as the holder of the perpetual unlevered (after tax) cash flow of the assets minus the perpetual tax-adjusted coupon flow plus the option to default, i.e., plus a put option to terminate the coupon payments in exchange for handing over the control rights to the debtholders. Therefore, I am first interested in the value of the perpetual cash flow.

Proposition 7 (value of the perpetual unlevered flow under perfect information) *The value function of the perpetual unlevered flow is $X_0(x)$ and $X_1(x)$ are given by*

$$\begin{aligned}X_0(x) &= \gamma_0 x \\ X_1(x) &= \gamma_1 x\end{aligned} \quad (26)$$

where the constants γ_0 and γ_1 are given by

$$\begin{aligned}\gamma_0 &= \frac{r-(\mu_h-\kappa_x\sigma_x)+\lambda_{hl}+\lambda_{lh}}{\lambda_{lh}(r-(\mu_h-\kappa_x\sigma_x))+(r-(\mu_l-\kappa_x\sigma_x))(r-(\mu_h-\kappa_x\sigma_x)+\lambda_{hl})} \\ \gamma_1 &= \frac{r-(\mu_l-\kappa_x\sigma_x)+\lambda_{hl}+\lambda_{lh}}{\lambda_{lh}(r-(\mu_h-\kappa_x\sigma_x))+(r-(\mu_l-\kappa_x\sigma_x))(r-(\mu_h-\kappa_x\sigma_x)+\lambda_{hl})}\end{aligned} \quad (27)$$

PROOF: The proof is provided in the Appendix.

I assume that equityholders have limited liability and are free to endogenously determine their optimal strategy of how to use the right to trigger bankruptcy. Obviously, the decision to default is conditioned on the currently prevailing growth rate, and if the firm is in the low growth scenario, equityholders will trigger bankruptcy earlier (i.e., at a higher level of the cash flow) than this is the case in the high growth scenario.

Proposition 8 (value of equity under perfect information) *Equity provides a flow of $x_t - (1-\tau)c$ to the claimholder as long as they maintain the coupon service to debtholders. Let $E_0(x)$ and $E_1(x)$ denote the value function of equity conditional on $\pi = 0$ and $\pi = 1$ respectively. The lower thresholds $\underline{x}_0 \geq \underline{x}_1$ denote the endogenous bankruptcy triggers. Then the value of equity is characterized by*

$$\begin{aligned}
E_0(x) &= E_{11}\Delta_1\, x^{\eta_1} + E_{12}\Delta_2\, x^{\eta_2} + \gamma_0 x - \frac{(1-\tau)c}{r} && \textit{for} \quad \underline{x}_0 \leq x \\
E_0(x) &= 0 && \textit{for} \quad x < \underline{x}_0 \\
E_1(x) &= E_{11}\, x^{\eta_1} + E_{12}\, x^{\eta_2} + \gamma_1 x - \frac{(1-\tau)c}{r} && \textit{for} \quad \underline{x}_0 \leq x \\
E_1(x) &= E_{13}\, x^{\beta_1} + E_{14}\, x^{\beta_2} + \frac{x}{r+\lambda_{hl}-\mu_h} - \frac{(1-\tau)c}{r+\lambda_{hl}} && \textit{for} \quad \underline{x}_1 \leq x < \underline{x}_0 \\
E_1(x) &= 0 && \textit{for} \quad x < \underline{x}_1
\end{aligned} \tag{28}$$

where η_1 and η_2 are the two negative roots of the characteristic 4th order polynomial, $Q(\eta)$, and β_1 and β_2 are the negative and the positive root of the characteristic quadratic polynomial, $R(\beta)$, both can be found in the Appendix. The constants Δ_1 and Δ_2 are determined by

$$\begin{aligned}
\Delta_1 &= \frac{r+\lambda_{hl}-\frac{1}{2}\eta_1^2\sigma_x^2 - \eta_1(\hat{\mu}_h - \frac{1}{2}\sigma_x^2)}{\lambda_{hl}} \\
\Delta_2 &= \frac{r+\lambda_{hl}-\frac{1}{2}\eta_2^2\sigma_x^2 - \eta_2(\hat{\mu}_h - \frac{1}{2}\sigma_x^2)}{\lambda_{hl}}
\end{aligned} \tag{29}$$

The four constants E_{11}, E_{12}, E_{13}, E_{14} are set such that the following boundary conditions are satisfied

$$\begin{aligned}
E_0(\underline{x}_0) &= 0 \\
\lim_{x\to\underline{x}_0^+} E_1(x) &= \lim_{x\to\underline{x}_0^-} E_1(x) \\
\lim_{x\to\underline{x}_0^+} \frac{\partial E_1(x)}{\partial x} &= \lim_{x\to\underline{x}_0^-} \frac{\partial E_1(x)}{\partial x} \\
E_1(\underline{x}_1) &= 0
\end{aligned} \tag{30}$$

The exit triggers are set in order to maximize the value of equity, i.e., to satisfy the first order condition

$$\begin{aligned}\lim_{x\to\underline{x}_0^+}\frac{\partial E_0(x)}{\partial x} &= 0\\ \lim_{x\to\underline{x}_1^+}\frac{\partial E_1(x)}{\partial x} &= 0\end{aligned}\tag{31}$$

PROOF: The proof is provided in the Appendix.

When the cash flow of the firm is above $\underline{x}_0$ then equityholders are running the firm independent of the prevailing growth scenario. In this case the expression for $E_0(x)$ contains the value of the perpetual cash flow ($\gamma_0 x$) minus the value of the tax-benefit adjusted coupon flow ($(1-\tau_p)c/r$) plus the value of the put option to abandon the firm. Unlike standard contingent claims models of the firm, the put option in the uncertain growth framework is the sum of two power functions (there exists two positive roots and two negative roots of the characteristic polynomial of the order of four, the two positive roots determine the behavior of call options, the two negative roots are used to characterize the put option inherent in right to abandon the firms operation). Analogously, the expression for $E_1(x)$ consists of the perpetual cash flow minus the value of the coupon flow plus the put under uncertain growth. Note that the exponents in the put option value are the same for low and for high growth and the constant $E_{0.}$ is proportional to $E_{1.}$. When the cash flow of the firm is between $\underline{x}_1$ and $\underline{x}_0$, then equityholders immediately trigger bankruptcy if growth is low, but they continue to run the firm if growth is high. Therefore, $E_0 = 0$ in this case. The system of differential equations (6) then collapses into a single ordinary differential equation that has the standard solution. The riskless interest rate r is increased by λ_{hl} (the transition intensity for a jump from the high growth to the low growth scenario) to account for the possible change in the growth rate (which is accompanied by immediate shutdown). The optimality conditions 31 require that at the optimal bankruptcy triggers, the value functions have to be smooth, otherwise the value of equity can be increased by changing the triggers.

Debtholders receive a coupon flow of c as long as the firm is operated by equityholders. According to Assumption 7 equityholders' decision to default on their obligation leads to a sell out of the firms assets for the value of the unlevered assets. Debtholders receive a fraction $(1-\phi)$ of the proceeds.

Proposition 9 (value of debt under perfect information) *Let $D_0(x)$ and $D_1(x)$ denote the value function of equity conditional on $\pi = 0$ and $\pi = 1$ respectively. The bankruptcy triggers $\underline{x}_0 \geq \underline{x}_1$ are set by equityholders. Then the value of debt is characterized by*

$$\begin{aligned}D_0(x) &= D_{11}\Delta_1\, x^{\eta_1} + D_{12}\Delta_2\, x^{\eta_2} + \frac{c}{r} && \textit{for} \quad \underline{x}_0 \leq x\\ D_0(x) &= (1-\phi)\gamma_0 x && \textit{for} \quad x < \underline{x}_0\\ D_1(x) &= D_{11}\, x^{\eta_1} + D_{12}\, x^{\eta_2} + \frac{c}{r} && \textit{for} \quad \underline{x}_0 \leq x\\ D_1(x) &= D_{13}\, x^{\beta_1} + D_{14}\, x^{\beta_2} + \frac{c}{r+\lambda_{hl}} && \textit{for} \quad \underline{x}_1 \leq x < \underline{x}_0\\ D_1(x) &= (1-\phi)\gamma_1 x && \textit{for} \quad x < \underline{x}_1\end{aligned}\tag{32}$$

where exponents η_1, η_2, β_1, β_2 *and the constants* Δ_1 *and* Δ_2 *are the same as in Proposition 8.*

The four constants D_{11}, D_{12}, D_{13}, D_{14} *are set such that the following boundary conditions are satisfied*

$$\begin{aligned} D_0(\underline{x}_0) &= (1-\phi)\gamma_0\underline{x}_0 \\ \lim_{x\to\underline{x}_0^+} D_1(x) &= \lim_{x\to\underline{x}_0^-} D_1(x) \\ \lim_{x\to\underline{x}_0^+} \frac{\partial D_1(x)}{\partial x} &= \lim_{x\to\underline{x}_0^-} \frac{\partial D_1(x)}{\partial x} \\ D_1(\underline{x}_1) &= (1-\phi)\gamma_1\underline{x}_1 \end{aligned} \tag{33}$$

PROOF: The proof is provided in the Appendix.

4.2 Exogenous Bankruptcy and Risk Neutrality with Respect to Belief Fluctuations

This subsection presents the analytical solution for the value of the firms debt and equity for the special case in which the market price for information risk, κ_π vanishes, i.e., where condition (17) is satisfied, and where the bankruptcy decision is exogenously given, such that bankruptcy occurs when some lower cash flow level $\underline{x}$ is hit for the first time. In this case investors do not ask for compensation for the volatility which is generated by the belief updating process. Since updating is positively correlated to the cash flow (the correlation of W_x and W_y is given by $dW_x dW_y = \sigma_y/\sigma_x$) this means that the positive contribution of the cash flow to κ_x has to be compensated by some negative price component associated with the innovation process W_s of the signal s. The assumed exogenous bankruptcy implies that equityholders (as well as debtholders) learn about the growth rate but are not allowed to set any action conditional on their information. The value functions of the relevant claims are functions of the current cash flow level x and the belief π and must satisfy the partial differential equation (15) in Proposition 3. Again it is assumed that κ_x, κ_s, and r are constants.

Proposition 10 (value of the perpetual unlevered flow when $\kappa_\pi = 0$) *When the market price for fluctuations in the belief vanishes,* $\kappa_\pi = 0$*, then the value of the unlevered perpetual cash flow x is a function of the cash flow and the belief given by*

$$X(x,\pi) = [\pi\gamma_1 + (1-\pi)\gamma_0]x, \tag{34}$$

where the constants γ_0 *and* γ_1 *are the same as in Proposition 7.*

PROOF: The proof is provided in the Appendix.

If $\kappa_\pi = 0$, the value of the perpetual cash flow is simply a convex linear-combination of the value functions $X_0(x)$ and $X_1(x)$ under perfect information. Fluctuations in the belief π lead only to a shift in the relative weights.

Proposition 11 **(value of equity when $\kappa_\pi = 0$)** *When the market price for fluctuations in the belief vanishes, $\kappa_\pi = 0$, and bankruptcy occurs when the cash flow hits the lower threshold $\underline{x}$. Then the value $E(x,\pi)$ is given by*

$$\begin{aligned} E(x,\pi) &= E_{11}[\pi+(1-\pi)\Delta_1]x^{\eta_1} \\ &\quad +E_{12}[\pi+(1-\pi)\Delta_2]x^{\eta_2}+X(x,\pi)-\frac{(1-\tau)c}{r} \quad \text{for} \quad \underline{x}\le x \\ E(x,\pi) &= 0 \quad \text{for} \quad x<\underline{x} \end{aligned} \tag{35}$$

The exponents η_1 and η_2 (the two negative roots of the characteristic polynomial) and the constants Δ_1 and Δ_2 are the same as in Proposition 8. The constants E_{11} and E_{12} are determined by the boundary condition

$$E(\underline{x},\pi) = 0 \quad \forall\pi \tag{36}$$

PROOF: The proof is provided in the Appendix.

Here it is again the fact that an ansatz which is linear in π solves the partial differential Equation (15). Please note that this is only true because $\kappa_\pi = 0$ and because the assumption of an exogenous bankruptcy preserves the linearity. If bankruptcy is endogenously determined, equityholders have the incentive to operate the firm the longer the more they are convinced that the firm is in the high growth state, thus, the endogenous bankruptcy threshold is a function of π and the linearity of E in π is violated. If $\kappa_\pi \neq 0$, then the distortion in the updating of the risk neutral belief destroys linearity. In both cases, one has to use numerical methods to solve for the equity value.

Proposition 12 **(value of debt when $\kappa_\pi = 0$)** *When the market price for fluctuations in the belief vanishes, $\kappa_\pi = 0$, and bankruptcy occurs when the cash flow hits the lower threshold $\underline{x}$. Then the value $D(x,\pi)$ is given by*

$$\begin{aligned} D(x,\pi) &= D_{11}[\pi+(1-\pi)\Delta_1]x^{\eta_1} \\ &\quad +D_{12}[\pi+(1-\pi)\Delta_2]x^{\eta_2}+\frac{c}{r} \quad \text{for} \quad \underline{x}\le x \\ D(x,\pi) &= (1-\phi)X(x,\pi) \quad \text{for} \quad x<\underline{x} \end{aligned} \tag{37}$$

The exponents η_1 and η_2 (the two negative roots of the characteristic polynomial) and the constants Δ_1 and Δ_2 are the same as in Proposition 8. The constants D_{11} and D_{12} are determined by the boundary condition

$$D(\underline{x},\pi) = (1-\phi)X(\underline{x},\pi) \quad \forall\pi \tag{38}$$

PROOF: The proof is provided in the Appendix.

Since the value of the unlevered perpetual cash flow is linear in the belief when $\kappa_\pi = 0$, an ansatz which is linear in π again solves the partial differential equation when bankruptcy occurs at a threshold $\underline{x}$ (equal for all π).

4.3 The General Case

The previous subsections discuss special cases which allow for an analytical solution of the valuation equation (15) which is a partial differential equation with respect to the the cash flow x and the belief π. The first is the case of perfect information, the second the case of endogenous bankruptcy and $\kappa_\pi = 0$. Whereas under perfect information the domain of the belief degenerates such that π can only take the two states $\pi = 0$ or $\pi = 1$, the second special case leads to value functions which depend linearly on the belief π. This linear dependency is destroyed if either default is endogenously determined or if $\kappa_\pi \neq 0$. Endogenous default leads to the fact that equityholders will default earlier (i.e., at a higher level of the cash flow) if π is low and they will maintain their obligations longer if π is high. Thus the default threshold is a function $\underline{x}(\pi)$. This threshold is expected to be decreasing in π, because higher expected growth gives an incentive to maintain the firm's obligations down to lower levels of x. The need to condition actions on the belief introduces value of information which implies that the value function is not longer a convex linear-combination of the extreme states $\pi = 0$ and $\pi = 1$.

The same is true when $\kappa_\pi \neq 0$. Then the dynamics of updating π in response to the observation of the cash flow and the signal under the risk neutral probability measure deviate from the objective dynamics of the Bayesian belief. The effect of this deviation of the risk neutral learning behavior can be seen best when valuing the perpetual unlevered cash flow. In this case there is no bankruptcy decision, hence, the effect of 'rational pessimism' can be demonstrated. The next subsection focuses on the perpetual flow.

Since the risk neutral dynamics of x and π are known (see Proposition 4), the valuation of the firm's stakes (i.e., different claims contingent on x and π) can be done on a two dimensional binomial tree. The only difficulty comes from the fact that π follows a mean reversion process with nonconstant volatility. Therefore, I use first the approach of Nelson and Ramaswamy (1990) to model the belief process π on the basis of a recombining tree. And second, I adapt the approach of Boyle, Evnine, and Gibbs (1989) to evaluate the bivariate contingent claims.[11]

The approach of Nelson and Ramaswamy (1990) to retain computational simplicity of a binomial tree is based on the transformation of the original belief process π_t into a process $z_t = Z(\pi_t)$ which is twice differentiable in π and has constant volatility.[12] Then the process z_t is approximated on a recombining tree and the belief process π is derived from $\pi_t = Z^{-1}(z_t)$. If certain conditions are satisfied, weak convergence of the approximation, i.e., convergency in distribution, to the continuous time process π_t is ensured when the step size of the discretization in the time dimension approaches zero.[13]

[11] In this procedure I assume that the market prices for risk κ_x and κ_s do not explicitly depend on time t, however, they possibly depend on the current belief π about growth.

[12] In general the transformation Z is time dependent, i.e., $Z = Z(\pi, t)$, however since the volatility of π does not explicitely depend on time, Z only depends on π.

[13] These conditions are discussed in Nelson and Ramaswamy (1990) and they are satisfied in this particular application.

Corollary 3 (recombining tree) *Take the transformation*

$$Z(\pi)=\int_{\frac{1}{2}}^{\pi}\frac{dp}{\sigma_\pi(p)}=\int_{\frac{1}{2}}^{\pi}\frac{\sigma_y}{(\mu_h-\mu_l)p(1-p)}dp=\frac{\sigma_y}{\mu_h-\mu_l}\ln\left(\frac{\pi}{1-\pi}\right). \tag{39}$$

Then the process $z_t=Z(\pi_t)$ has constant volatility of 1 *and follows the dynamics*

$$\begin{aligned} dz &= \underbrace{\left[\hat{\mu}_\pi\frac{dZ(\pi)}{d\pi}+\frac{1}{2}\sigma_\pi^2\frac{d^2Z(\pi)}{d\pi^2}\right]}_{\hat{\mu}_z}dt+dW_y^{\mathcal{F}} \\ z_0 &= Z(\pi_0) \end{aligned} \tag{40}$$

where σ_π is the volatility and $\hat{\mu}_\pi$ is the risk neutral drift of the belief (see Proposition 4) given by

$$\begin{aligned} \sigma_\pi &= \frac{(\mu_h-\mu_l)\pi(1-\pi)}{\sigma_y} \\ \hat{\mu}_\pi &= -\pi\lambda_{hl}+(1-\pi)\lambda_{lh}-\kappa_\pi\sigma_\pi \end{aligned} \tag{41}$$

If z_t is approximated on a binomial tree, then the belief process results form applying the inverse, $\pi_t=Z^{-1}(z_t)$, that is given by

$$Z^{-1}(z)=\frac{1}{1+\exp\left\{-\left(\frac{\mu_h-\mu_l}{\sigma_y}\right)z\right\}}. \tag{42}$$

PROOF: The proof is provided in the Appendix.

Due to the volatility of 1 the binomial jumps in the tree for z_t have the magnitude of $\pm\sqrt{h}$ if h is the step size in the discretization with respect to time. The probability for an up-move and the down move are set in order to guarantee an expected drift of $\hat{\mu}_z h$. Therefore, these probabilities are state dependent. The only complication comes from the fact that $\sigma_\pi(\pi=0)=\sigma_\pi(\pi=1)=0$, however, one can overcome this difficulty by introducing multiple jumps in the neighborhood of $\pi=0$ and $\pi=1$ (the general idea of this approach is also discussed in the paper of Nelson and Ramaswamy (1990)).

Since the process z_t is driven by the Brownian motion $W_y^{\mathcal{F}}$ (the innovation process that drives the dynamics of the minimum variance compound process y_t) the correlation of z_t to the cash flow x_t is such that $\frac{dx}{x}\,dz_t=\sigma_y\,dt$.

The approach of Boyle, Evnine, and Gibbs (1989) to generate a two-dimensional binomial tree has to be adapted slightly because the drift rate of x_t (the first dimension of the tree) depends on z_t (the second dimension), or more exactly the risk neutral drift $\hat{\mu}_x$ is given by

$$\hat{\mu}_x=\pi\mu_h+(1-\pi)\mu_l-\kappa_x\sigma_x=Z^{-1}(z)\mu_h+[1-Z^{-1}(z)]\mu_l-\kappa_x\sigma_x.$$

The tree-approach of course depends on the terminal condition that is applied at some time T, however for $T\to\infty$ the explicit time dependency and the dependency on the terminal condition vanishes.

4.3.1 The Value of the Perpetual Unlevered Cash Flow

The volatility of the claim on the perpetual unlevered flow is not constant over time, see Proposition 2. When investors have precise knowledge about the growth state, it is lowest. The claim value is perfectly correlated to the cash flow process in this case. If the knowledge is less precise (if π is in the neighborhood of 0.5), there is excess volatility coming from the fluctuation in the beliefs about the growth rate. This additional uncertainty is driven by the minimum variance compound process y, which results in the fact that the claim value is no longer perfectly correlated to x. Therefore, the price for risk associated with the claim is not constant over time, even if κ_x and κ_s are.

This has significant impact on the pricing of claims contingent on the firm's cash flow (see Proposition 3) and, furthermore, to a deviation of the risk neutral updating of the beliefs from what the Bayesian learning rule under the objective probability measure advises. If risk premia associated with the cash flow and the signal are nonnegative in the sense of Condition (18), then investors exhibit 'rational pessimism', that is, they put more weight on the low growth scenario than the objective Bayesian learning does. If Condition (18) is violated then some component of risk in x or s has negative price. Then growing volatility due to reduced knowledge of the growth state may result in a reduction of the total risk premium, in which case the risk neutral learning is 'rationally optimistic'.

Proposition 13 (value of the perpetual unlevered cash flow) *The value X of the perpetual unlevered cash flow generated by the firm's assets is linear in the level of the cash flow, i.e.,*

$$X(x,\pi) = xX(1,\pi).$$

However, $X(x,\pi)$ is not necessarily a π-weighted convex combination of the extreme values $X(x,0)$ and $X(x,1)$. This is only the case if risk neutral learning is equivalent to objective learning, i.e., if $\kappa_\pi = 0$, see the special case.

Figure 2 shows the value of the claim on the perpetual unlevered cash flow at $x = 1$ for the base case parameters given in Table 2 and different signal characteristics. If $\kappa_\pi > 0$ then investors behavior is 'rationally pessimistic'. They value the claim as if they were reluctant to gain confidence in the high growth scenario and as if they put more weight to the low growth scenario than Bayes' rule advises. Investors seem to trust more in the negative news than in the positive. This results in lower valuation of the perpetual flow and in a convex shape of the value function with respect to π. Furthermore, an increase in the information quality of the signal, i.e., a lower σ_s, does not not necessarily result in higher valuation. Compare the value functions for $\sigma_s = 1.0$, $\sigma_s = 0.1$ in Figure 2 both with $\kappa_s = 0.5$ and $\rho = 0.0$. This is because different information content in the signal changes the weights w_x and w_s in the minimum volatility compound process y_t on which learning is conditioned. Therefore, higher information quality may lead to more pronounced pessimistic behavior of the investors, see Figure 1(d) for the dependency of the expected value of the stationary risk neutral belief density on the volatility of the signal. If the

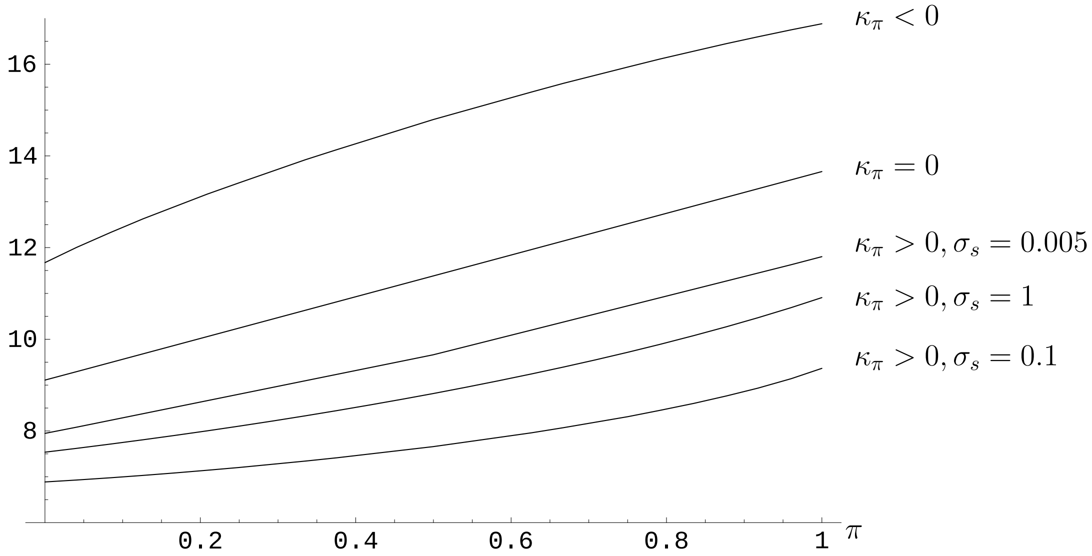

Figure 2: The value of the perpetual unlevered cash flow at $x = 1$ as a function of the belief π for the base case parameters (see Table 2) and different characteristics of the signal s. For $\kappa = 0$ (the curve is computed for $\kappa_s = 0.1$, $\rho = 0.8$ and $\sigma_s = 0.1786$), the value is identical to the value under perfect information. When $\kappa > 0$ (the curves are computed for $\kappa_s = 0.5$, $\rho = 0.0$), investors behave pessimisticly, the value is below the value under perfect information and convex in the belief. The convergency to the value function under perfect information is not necessarily monotone in the informativeness of the signal σ_s. When $\kappa_\pi < 0$ (the curve is computed for $\kappa_s = 0.1$, $\rho = 0.8$ and $\sigma_s = 0.1$), investors valuation is optimistic, the value is above the value under perfect information and concave in the belief.

excess volatility coming from belief fluctuations is purely idiosyncratic, i.e., if $\kappa_\pi = 0$ which means that investors do not require compensation for the uncertainty in the growth rate, then investors valuation is the same as under perfect information. The result of the numerical procedure matches the analytical results obtained in the previous two subsections. If the risk premium associated with belief fluctuations is negative, i.e., $\kappa_\pi > 0$, investors behave 'rationally optimistic'. In this case they seem to belief more in the positive signals than in the negative and value the claim even higher than under perfect information.

Table 2: Parameters for the base case.

drift rate in the high growth scenario	μ_h	0.185
drift rate in the low growth scenario	μ_l	0.065
riskless rate of interest$r$	0.1	
instantaneous transition density from high to low growth	λ_{hl}	0.1
instantaneous transition density from low to high growth	λ_{lh}	0.1
volatility of the cash flow	σ_x	0.25
price for risk associated with W_x	κ_x	0.5

4.3.2 The Value of the Equity

The uncertainty in the growth rate is expected to have significant effect on the value of the firms equity for two reasons. First, equityholders face the threat of inefficient default when determining the default-threshold. The decision to maintain the firms obligation in times of low cash flow depends crucially on the expectation on the drift rate, and therefore, it depends on what equityholders expect to learn about growth in the near future. Second, the deviation of the learning dynamics under the risk neutral measure from the objective Bayesian rule influences the valuation of the underlying asset (i.e., the cash flow x, see the previous section). Given a certain capital structure, equityholders receive a cash flow of $x-(1-\tau)c$ as long as they operate the firm. The firms equity is therefore interpreted as the value of the perpetual unlevered flow (i.e., $X(x,\pi)$) minus the tax adjusted perpetual coupon flow (i.e., $(1-\tau)c/r$) plus an American put option that allows equityholders to hand over the assets of the firm to the debtholders in exchange for terminating coupon payment, i.e., equityholders have limited liability. For numerical computations, the terminal condition at some T is given by $\max\{0\ ,X(x,\pi)-\frac{(1-\tau)c}{r}\}$. For large $T\to\infty$ the stationary solution is the time independent equity value.

Figure 3 shows the typical shape of the equity value function when $\kappa_\pi > 0$. Equity value as a function of the cash flow and the belief is shown in the surface plot in Figure 3(a). Figure 3(b) shows the equity as a function of the cash flow at the boundaries $\pi = 0$ and $\pi = 1$ (when knowledge about growth is perfect) together with the asymptotes $X(x,0)-(1-\tau)c/r$ and $X(x,1)-(1-\tau)c/r$. The equity exhibits the familiar convex shape that asymptotically converges to the value of the underlying asset (the perpetual flow minus the perpetual obligations from the capital structure) and it touches smoothly the abscissa, which characterizes the threshold where equityholders optimally default). Apparently, equity value is lower when investors are convinced that growth is low, and consequently, the default threshold is higher in this case. The contour plot in Figure 3(c) shows isoquants of the equity value. One can see the shape of the default threshold $\underline{x}(\pi)$ (which is the lowest isoquant at the left hand side). The convex shape of the default threshold expresses value of waiting. So if the cash flow is low and knowledge about growth is imprecise, there is an incentive to stay in and operate the firm and try to aquire better information about the growth state in order to avoid inefficient default. This incentive vanishes when the belief drifts towards $\pi = 0$ and is more pronounced when the belief drifts towards $\pi = 1$, which leads to the convex shape. It has to be mentioned that the terminal condition $\max\{0\ ,X(x,\pi)-\frac{(1-\tau)c}{r}\}$ creates a concave default threshold when $\kappa_\pi > 0$, so if information quality is poor and κ_π is very large, then there is only little chance that π increases, thus, the exit threshold can also be concave. 3(d) shows equity value as a function of π for certain values of the cash flow near the default threshold. One can see that equity is also convex in π. If the cash flow exceeds only slightly $\underline{x}(1)$, then equityholders optimally run the firm only if they are convinced that the firm is in the high growth scenario. For growing cash flows equityholders are by and by willing to operate the firm also when the belief turns towards lower values. If the cash flow exceeds $\underline{x}(0)$, then equityholders operate the firm independent of their belief about growth.

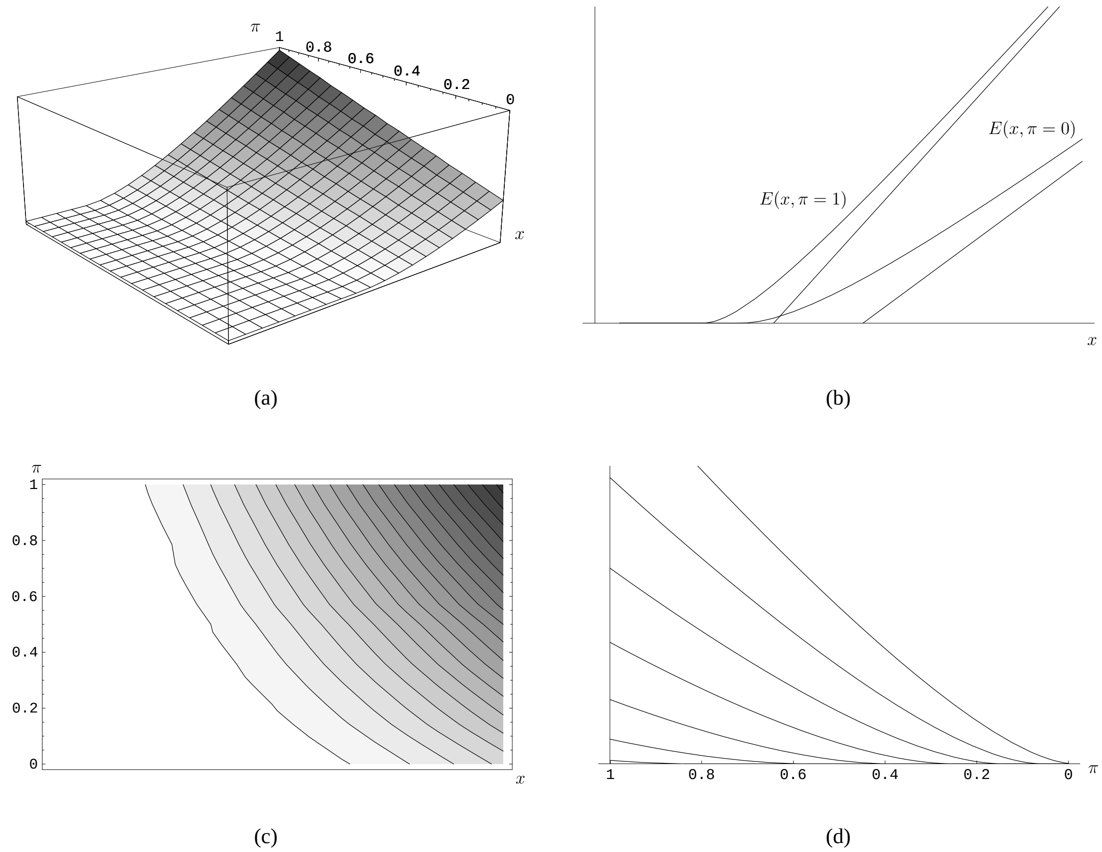

Figure 3: The typical shape of the value of equity in a changing environment. (a) Equity value as a function of x and π. (b) Equity value at the boundaries $\pi = 0$ and $\pi = 1$ as a function x. (c) Isoquants of the value of equity in a contour plot, the optimal default threshold $\underline{x}(\pi)$ is the lowest isoquant on the left-hand side. (c) The value of equity as a function of π for different x.

The dependency of the default threshold $\underline{x}(\pi)$ on the signal quality is demonstrated in Figure 4. The plotted thresholds are computed for the base case parameters and for uncorrelated signals with idiosyncratic noise (i.e., $\kappa_s = 0$). The Figure also shows the exit threshold under perfect information (the solid straight line indicates $\underline{x}_1$, the default threshold when growth is high and information is perfect, the dashed line indicates $\underline{x}_0$, the default threshold when growth is low and information is perfect). It can be seen that for $\pi > 0$ increasing informativeness of the signal leads to convergency of $\underline{x}(\pi)$ to $\underline{x}_1$, and for $\pi = 0$ it converges to $\underline{x}_0$. That means, if the informativeness of the signal is nearly perfect and π is inside the interval $(0, 1)$, then waiting only for a moment will immediately reveal the current state. Thus, whenever $\pi > 0$ and $\underline{x}(\pi) > \underline{x}_1$ equityholders will hang on and wait to get knowledge about the state and make the default decision thereafter. Again, the convergency is not necessarily monotone in the informativeness of the signal, i.e., lower σ_s does not necessarily imply that equityholders default decision moves closer to the default decision under perfect information. However, due to the assumption that the noise in the signal is completely idiosyncratic, it is monotone in this figure.

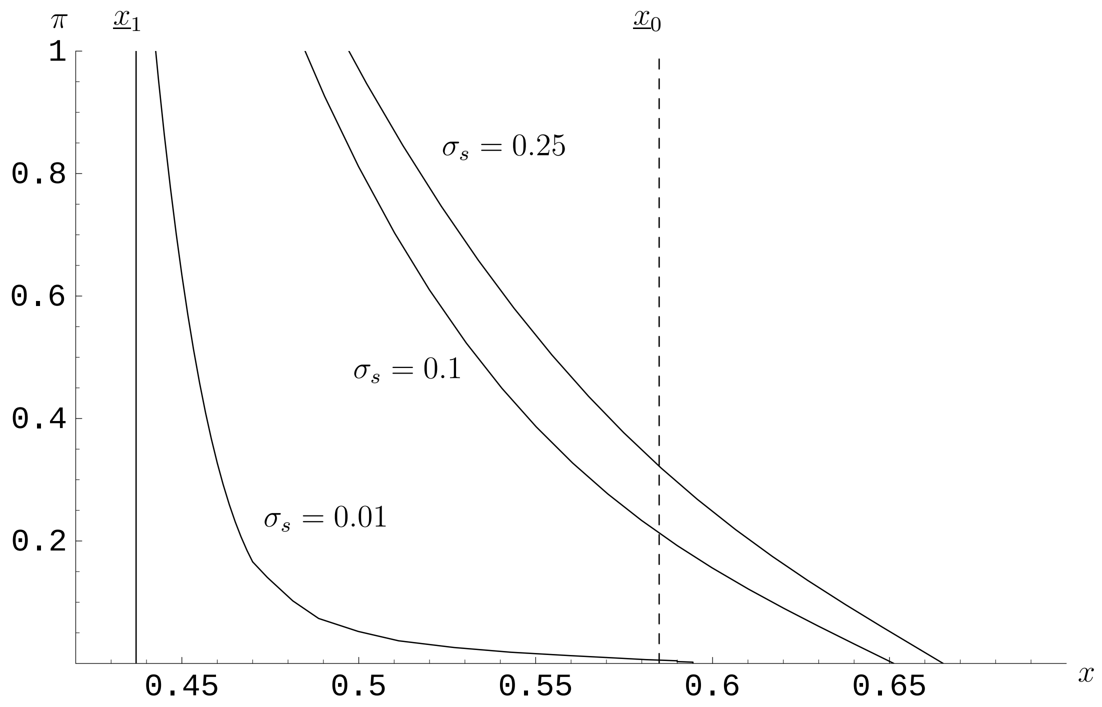

Figure 4: Optimal default thresholds $\underline{x}(\pi)$ for different information quality of the signal (for the base case parameters under the assumption $\rho = 0$ and $\kappa_s = 0$). The optimal exit thresholds under perfect information ($\underline{x}_0$ and $\underline{x}_1$) are indicated by vertical lines.

4.3.3 The Value of the Debt

Suppose a given bond that obliges equityholders to provide a coupon flow of c to debtholders. The value of the perpetual flow is simply c/r. However, equityholders have limited liability. If the cash flow hits the boundary $\underline{x}(\pi)$ for the first time, they default on their obligations (see the previous section on this topic) and hand the productive assets over to debtholders in exchange for a cancellation of the coupon payment. Debtholders are assumed to be not able to operate the firm, but immediately sell the assets such that—after paying the costs associated with bankruptcy—they receive a fraction ϕ of the value of the unlevered assets. debtholders

Figure 5 shows the typical shape of a bond value function. For illustrative purposes the debt value is computed with $\phi = 1$, i.e., the productive assets are worthless in the case of bankruptcy. Debt value as a function of the cash flow and the belief is shown in the surface plot in Figure 5(a). Figure 5(b) shows the value of debt as a function of the cash flow at the boundaries $\pi = 0$ and $\pi = 1$ (when knowledge about growth is perfect) together with the common asymptote c/r. Both extreme cases exhibit the familiar concave shape in x. Apparently, the value of debt is lower in the low growth scenario and due to the lower drift rate, the convergency to the asymptote is slower. The contour plot in Figure 5(c) shows isoquants of the value of debt which exhibit the convex shape of equityholders' default threshold $\underline{x}(\pi)$. Figure 5(d) shows debt value as a function of π for certain values of the cash flow near the default threshold. One

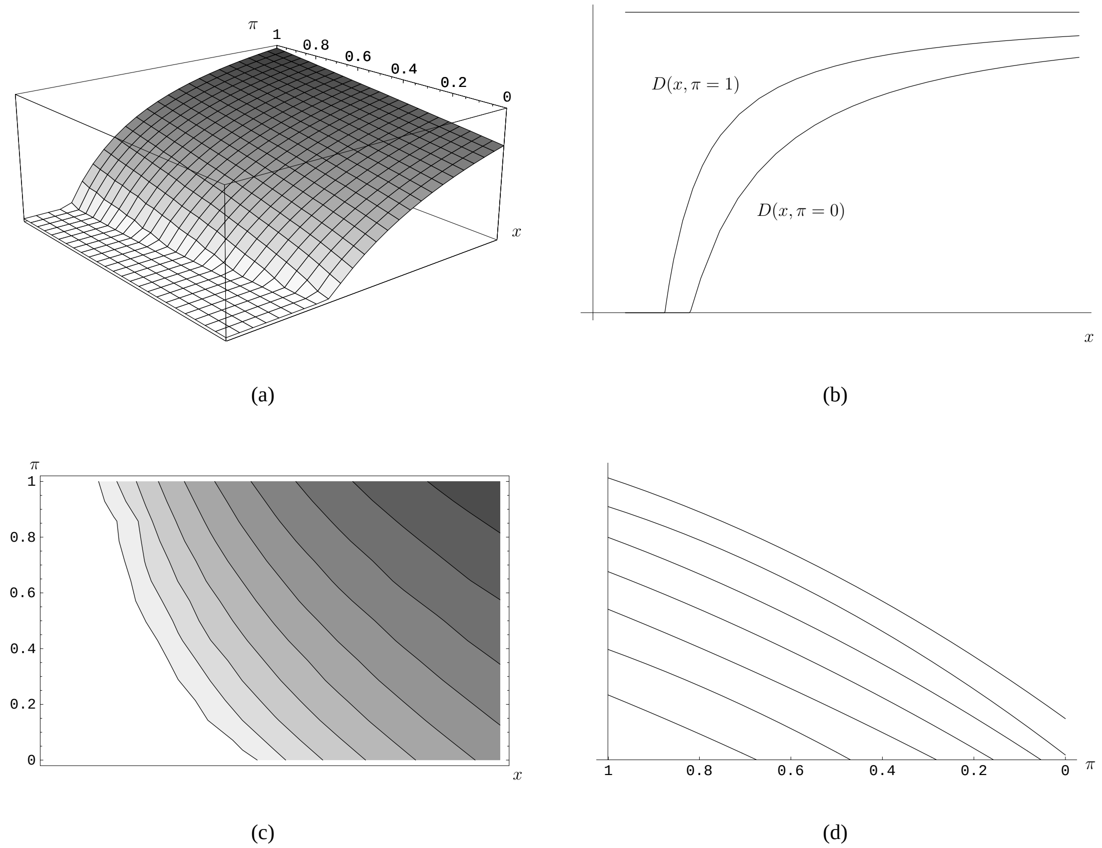

Figure 5: The typical shape of the value of debt in a changing environment. (a) Debt value as a function of x and π. (b) Debt value at the boundaries $\pi = 0$ and $\pi = 1$ as a function x. (c) Isoquants of the value of debt in a contour plot. (c) The value of debt as a function of π for different x.

can see that debt is also concave in π.

5 Optimal Capital Structure Choice

The final issue to be addressed is the paper is the question about the optimal capital structure of a firm that operates in an uncertain environment. In the previous section equity and debt are valued when the coupon c is given. Now the owners of the firm (all equity) want to lever it optimally (i.e., they want to maximize total firm value) under the assumption that there is a tax advantage of debt (see Assumption 6). From the point of view of the single firm the market conditions (i.e., the prices for risk κ_x and κ_s) are exogenously given. The distortion of the belief updating process analyzed in this paper has significant effects on the capital structure choice.

Figure 6 illustrates this effect for the base case parameters and noncorrelated, idiosyncratic signals of different information quality. The tax advantage of debt is determined by the assumption of $\tau = 0.1$, the costs of bankruptcy are assumed to be $\phi = 25\%$. Figure 6(a) and 6(b) show the optimal initial value

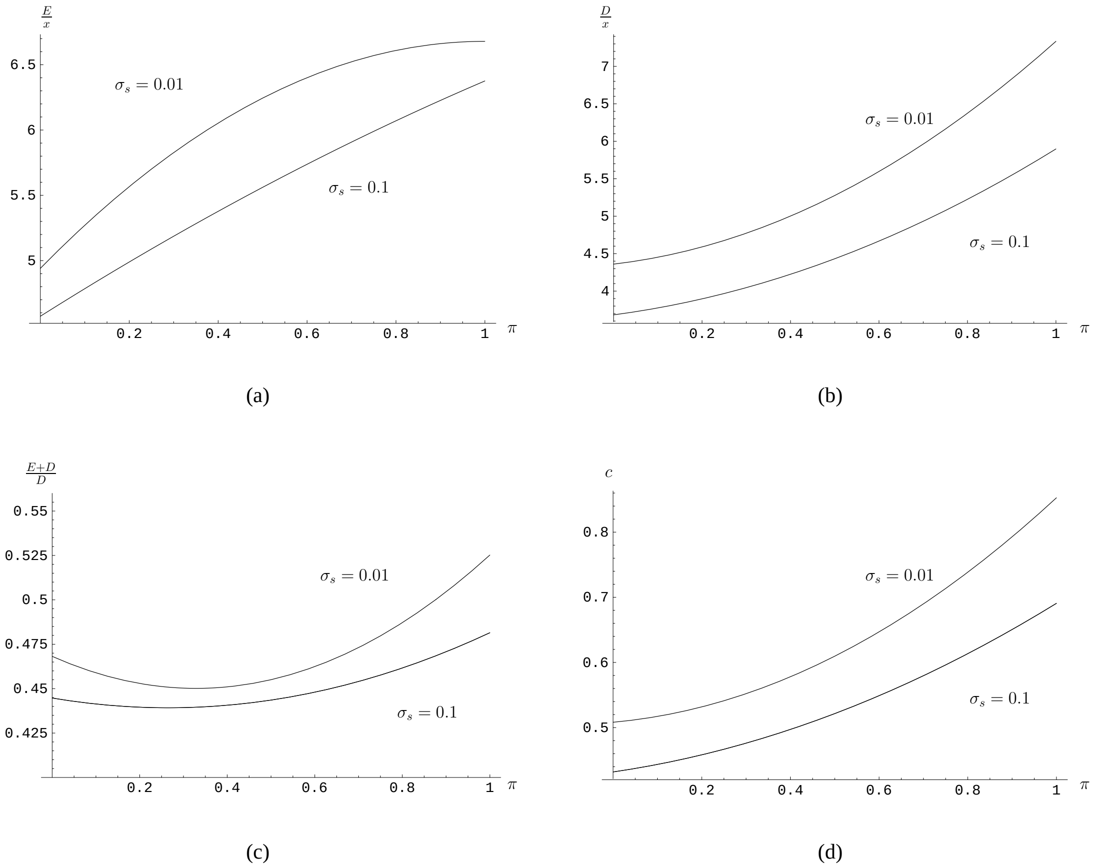

Figure 6: Effect of the signal quality on the optimal capital structure choice when the signal is uncorrelated and the signal noise is completely idiosyncratic, $\tau = 0.1$, $\phi = 0.75$. (a) The initial value of equity per unit of cash flow when the capital structure is chosen optimally (b) The initial value of debt per unit of cash flow (c) The optimal initial leverage ratio (d) The optimally contracted coupon

equity and debt per unit of cash flow. We see that both, equity value and debt value, are higher in the case of better signal quality. Thus, under given prices for risk, lower signal quality results in lower debt capacity of the firm, which is a indirect consequence of investors pessimistic expectation about the firm's future, which is (when the signal is noncorrelated and fluctuations in the signal are idiosyncratic) the more pronounced the lower the information quality is. Equity value is concave in the belief π and debt value is convex in π and the curvature is more pronounced if signal quality is better. (Please note that after the bond is issued, equity is convex and debt is concave in the belief). Concave equity and convex debt lead to u-shaped optimal leverage ratio, see Figure 6(c). Again, the curvature is more pronounced in the case of higher signal precision. Finally, Figure 6(c) plots the optimally contracted coupon, which shows again, the positive dependency of debt capacity on information quality.

6 Conclusion

This paper develops a contingent claims model of the firm which operates in an uncertain growth environment. In a partial equilibrium framework it is shown that the risk neutral dynamics of investors belief about the cash flow's growth rate deviates from the objective Baysian rule when learning is conditioned on the realizations of the cash flow and an informative signal. If risk premia are positive (in a strong sense) then investors exhibit 'rational pessimism', such that they put more weight to the low growth scenario than the objective Bayesian rule advises. In two special cases there are analytical solutions available for the value function of debt and equity. For the general case there is a numerical procedure presented which operates on a two dimensional tree and uses the approach of Nelson and Ramaswamy (1990) to guarantee computationally simple tree structures. The effect of investors pessimism on corporate decision making—first of all the optimal default threshold—is studied and the effect on the value of debt and equity is illustrated. Finally, the optimal initial capital structure is studied.

Possible extensions of the paper are studies that make the firm's capital structure more flexible. This can be done by introducing debt renegotiation in order to prevent expensive bankruptcy. Or one can add certain call features to the debt contract which allow the firm to increase dynamically its debt level by calling existing bonds and issuing new contacts with higher face value.

Appendix

A.1 Proof of Corollary 1

Using Ito's Lemma and the fact that $\mathbf{1}'w = 1$ (i.e., the sum of the weights is 1), the dynamics of y_t can be derived as

$$\begin{aligned} dy &= \frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial x}dx + \frac{1}{2}\frac{\partial^2 y}{\partial^2 x^2}(dx)^2 + \frac{\partial y}{\partial s}ds + \frac{1}{2}\frac{\partial^2 y}{\partial s^2}(ds)^2 + \frac{\partial^2 y}{\partial x \partial s}dx\, ds \\ &= \mu_t\, dt + w_x \sigma_x (dW_x)_t + w_s \sigma_s (dW_s)_t \end{aligned}$$

Thus, y follows a Brownian motion with drift μ_t and instantaneous variance $w'Vw$. Rewriting Assumptions 1 and 4 yields

$$\begin{aligned} \sigma_x (dW_x)_t &= \frac{dx_t}{x_t} - \mu_t dt, \\ \sigma_s (dW_s)_t &= \frac{ds_t}{s_t} - \mu_t dt. \end{aligned}$$

Since the weights sum to one, we get

$$dy = w_x \frac{dx}{x} + w_s \frac{ds}{s}.$$

Choosing the weights in order to minimize this variance means to solve

$$\begin{aligned} w'Vw &\rightarrow \min, \\ s.t. \quad \mathbf{1}'w &= 1, \end{aligned}$$

which yields the expressions for w and σ_y that are stated in the corollary.

The diffusion term in dy can then be written as

$$\begin{aligned} \sigma_y (dW_y)_t &= w_x \sigma_x (dW_x)_t + w_s \sigma_s (dW_s)_t \\ &= \frac{1}{\mathbf{1}'V^{-1}\mathbf{1}} \mathbf{1}'V^{-1} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_s \end{pmatrix} \begin{pmatrix} dW_x \\ dW_s \end{pmatrix}_t \\ &= \sigma_y \underbrace{\left[\sigma_y \mathbf{1}'V^{-1} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_s \end{pmatrix} \begin{pmatrix} dW_x \\ dW_s \end{pmatrix}_t \right]}_{(dW_y)_t}. \end{aligned}$$

A.2 Proof of Corollary 2

Bayesian inference in continuous time utilizes the fact that the conditional instantaneous transition density of the observed processes differs depending on the current value of the hidden process. Thus, a particular realization of dx and ds gives rise to an updated belief of the hidden process, i.e., the growth rate. The discriminating power of the Bayesian theorem depends only on the relative deviation of the densities conditional on different values of μ.

If $f(x_1, s_1, x_2, s_2)$ denotes the joint transition density of x and s for moving from x_1, s_1 to x_2, s_2 during dt, then

The joint conditional instantaneous density of dx/x and ds/s is normal. If $f(\delta_x dt, \delta_s dt)$ denotes the instantaneous density that $dx/x = \delta_x dt$ and $ds/s = \delta_s dt$, then the densities conditional on $\mu_t = \mu_h$ and $\mu_t = \mu_l$ respectively are given by

$$\begin{aligned} f(\delta_x dt, \delta_s dt | \mu_t = \mu_h) &= \frac{1}{2\pi\sigma_x\sigma_s dt\sqrt{1-\rho^2}} \exp\left\{ -\frac{dt\left[\left(\frac{\delta_x-\mu_h}{\sigma_x}\right)^2 - 2\rho\left(\frac{\delta_x-\mu_h}{\sigma_x}\right)\left(\frac{\delta_s-\mu_h}{\sigma_s}\right) + \left(\frac{\delta_s-\mu_h}{\sigma_s}\right)^2 \right]}{2(1-\rho^2)} \right\}, \\ f(\delta_x dt, \delta_s dt | \mu_t = \mu_l) &= \frac{1}{2\pi\sigma_x\sigma_s dt\sqrt{1-\rho^2}} \exp\left\{ -\frac{dt\left[\left(\frac{\delta_x-\mu_l}{\sigma_x}\right)^2 - 2\rho\left(\frac{\delta_x-\mu_l}{\sigma_x}\right)\left(\frac{\delta_s-\mu_l}{\sigma_s}\right) + \left(\frac{\delta_s-\mu_l}{\sigma_s}\right)^2 \right]}{2(1-\rho^2)} \right\}. \end{aligned}$$

If learning is conditioned on the realization of y the relevant density is $g(dy = \delta_y\, dt)$ which is normal.

The conditional densities are

$$
\begin{aligned}
g(\delta_y\, dt|\mu_t=\mu_h) &= \frac{1}{\sqrt{2\pi}\sigma_y^2\, dt}\exp\left\{-\frac{dt}{2}\left(\frac{\delta_y-\mu_h}{\sigma_y}\right)^2\right\}\\
g(\delta_y\, dt|\mu_t=\mu_l) &= \frac{1}{\sqrt{2\pi}\sigma_y^2\, dt}\exp\left\{-\frac{dt}{2}\left(\frac{\delta_y-\mu_l}{\sigma_y}\right)^2\right\}
\end{aligned}
$$

Taking the relative deviation of the joint conditional densities of dx/x and ds/s and substituting the expressions for w_s, w_s, d_y and σ_y from Corollary 1 yields

$$
\begin{aligned}
\frac{f(\delta_x dt,\delta_s dt|\mu_t=\mu_l)}{f(\delta_x dt,\delta_s dt|\mu_t=\mu_h)} &= \exp\left\{\frac{dt(m_h-m_l)}{(1-\rho^2)\sigma_x^2\sigma_s^2}\times\right.\\
&\quad\left.\left[\frac{m_h+m_l}{2}(\sigma_x^2-2\rho\sigma_x\sigma_s+\sigma_s^2)-\delta_x(\sigma_s^2-\rho\sigma_x\sigma_s)-\delta_s(\sigma_x^2-\rho\sigma_x\sigma_s)\right]\right\}\\
&= \exp\left\{\frac{dt(m_h-m_l)}{2\sigma_y^2}\left[m_h+m_l-2(w_x\delta_x+w_s\delta_s)\right]\right\}\\
&= \frac{g((w_x\delta_x+w_s\delta_s)\, dt|\mu_t=\mu_l)}{g((w_x\delta_x+w_s\delta_s)\, dt|\mu_t=\mu_h)}.
\end{aligned}
$$

Thus, conditioning the learning on the observation of the two processes x_t and s_t leads to exactly the same belief than learning from observing the compound process y_t. Therefore, the inference problem is reduced to a one dimensional problem and, thus, the nonlinear filter theorem of Liptser and Shiryaev (2000), Chapter 9, can directly be applied.

A.3 Proof of Proposition 1

Substituting the definitions of $dW_x^{\mathcal{F}}$, $dW_s^{\mathcal{F}}$, and $dW_y^{\mathcal{F}}$ into Equation (13) yields

$$
\begin{aligned}
\begin{pmatrix} dx/x \\ ds/s \\ dy \end{pmatrix}_t &= [\pi_t\mu_h+(1-\pi_t)\mu_l]\begin{pmatrix}1\\1\\1\end{pmatrix}dt\\
&\quad+\begin{pmatrix}\sigma_x & 0 & 0\\ 0 & \sigma_s & 0\\ 0 & 0 & \sigma_y\end{pmatrix}\begin{pmatrix}\frac{1}{\sigma_x}\left[\frac{dx}{x}-(\pi_t\mu_h+(1-\pi_t)\mu_l)dt\right]\\ \frac{1}{\sigma_s}\left[\frac{ds}{s}-(\pi_t\mu_h+(1-\pi_t)\mu_l)dt\right]\\ \frac{1}{\sigma_y}\left[dy-(\pi_t\mu_h+(1-\pi_t)\mu_l)dt\right]\end{pmatrix}\\
&= \begin{pmatrix} dx/x \\ ds/s \\ dy \end{pmatrix}_t
\end{aligned}
$$

The prove the correlation between x and y, one can use the fact that y is the minimum volatility compound process created from x and s (see Corollary 1). Further noise reduction using x and y cannot be possible, because we have already shown that y carries the entire information about μ_t that is available. Applying noise reduction to x and y must again result in a volatility of σ_y, otherwise there is a contradiction to Corollary 2. Let ρ_{xy} denote the correlation coefficient between $dW_x^{\mathcal{F}}$ and $dW_y^{\mathcal{F}}$, then we have

$$\sigma_y^2 = \frac{(1-\rho_{xy})\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2-2\rho_{xy}\sigma_x\sigma_y}$$

Solving for ρ_{xy} gives σ_y/σ_x. The same argument determines the correlation between s and y.

A.4 Proof of Proposition 2

This Proposition follows from applying Ito's Lemma after series expansion of the value function $F(x,\pi,t)$ and using the $\mathcal{F}$-dynamics of the cash flow x and of the Bayesian belief π_t (see Definition 3 and Corollary 2). Furthermore, the fact that $(dW_x)_t(dW_y)_t = \frac{\sigma_y}{\sigma_x}dt$ is used.

$$\begin{aligned} dF &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2 + \frac{\partial F}{\partial \pi}d\pi + \frac{1}{2}\frac{\partial^2 F}{\partial \pi^2}(d\pi)^2 + \frac{\partial^2 F}{\partial x \partial \pi}dx\, d\pi + o(dt) \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}(x(\pi\mu_h + (1-\pi)\mu_l)dt + x\sigma_x(dW_x^{\mathcal{F}})_t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}x^2\sigma_x^2 dt \\ &\quad + \frac{\partial F}{\partial \pi}([-\pi\lambda_{hl} + (1-\pi)\lambda_{lh}]dt + \frac{(\mu_h-\mu_l)\pi_t(1-\pi_t)}{\sigma_y}(dW_y^{\mathcal{F}})_t) + \frac{1}{2}\frac{\partial^2 F}{\partial \pi^2}\frac{(\mu_h-\mu_l)^2\pi_t^2(1-\pi_t)^2}{\sigma_y^2}dt \\ &\quad + \frac{\partial^2 F}{\partial x \partial \pi}x\sigma_x\frac{(\mu_h-\mu_l)\pi_t(1-\pi_t)}{\sigma_y}\frac{\sigma_y}{\sigma_x}dt + o(dt). \end{aligned}$$

Dividing this equation by F, the coefficients of dt, $(dW_x^{\mathcal{F}})_t$ and $(dW_y^{\mathcal{F}})_t$ yield the expressions for α_F, σ_{Fx}, σ_{R_π} in the proposition.

A.5 Proof of Proposition 3

The compound innovation process dW_y is defined in corollary 1, which states that the contribution of dW_x to dW_y is

$$\sigma_y \mathbf{1}'V^{-1}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_s \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sigma_y \mathbf{1}'V^{-1}\begin{pmatrix} \sigma_x \\ 0 \end{pmatrix}$$

and that the contribution of dW_s to dW_y is

$$\sigma_y \mathbf{1}'V^{-1}\begin{pmatrix} 0 \\ \sigma_s \end{pmatrix}$$

Therefore, the dynamics of the claim F (see Proposition 2) can be rewritten as

$$\begin{aligned}\frac{dF_t}{F} &= \alpha_F\,dt + \left[\sigma_{Fx} + \sigma_{F\pi}\sigma_y \mathbf{1}'V^{-1}\begin{pmatrix}\sigma_x\\0\end{pmatrix}\right](dW_x^{\mathcal{F}})_t\\ &\quad +\sigma_{F\pi}\sigma_y\mathbf{1}'V^{-1}\begin{pmatrix}0\\\sigma_s\end{pmatrix}(dW_s^{\mathcal{F}})_t,\end{aligned}$$

From the theory of incomplete markets (see e.g., Björk (1998) for a discussion of this topic) and Assumption 8 it follows that the value function of the claim must satisfy

$$\begin{aligned} rF &= kx_t + d + F\cdot(\alpha_F)_t - F\cdot(\kappa_x)_t\left[(\sigma_{Fx})_t + (\sigma_{F\pi})_t\sigma_y\mathbf{1}'V^{-1}\begin{pmatrix}\sigma_x\\0\end{pmatrix}\right]\\ &\quad -F\cdot(\kappa_s)_t(\sigma_{F\pi})_t\sigma_y\mathbf{1}'V^{-1}\begin{pmatrix}0\\\sigma_s\end{pmatrix}\\ &= kx_t + d + F\cdot(\alpha_F)_t - F\cdot(\kappa_x)_t(\sigma_{Fx})_t - F\cdot(\sigma_{F\pi})_t\sigma_y\mathbf{1}'V^{-1}\begin{pmatrix}(\kappa_x)_t\sigma_x\\(\kappa_s)_t\sigma_s\end{pmatrix}\\ &= kx_t + d + F\cdot(\alpha_F)_t - F\begin{pmatrix}\kappa_x\\\kappa_\pi\end{pmatrix}_t'\begin{pmatrix}\sigma_{Fx}\\\sigma_{F\pi}\end{pmatrix}_t.\end{aligned}$$

Solving for $\kappa_\pi = 0$ gives Condition (17). Let $\tilde{\kappa}_s(\sigma_s)$ denote the price for W_s-risk as a function of σ_s that makes κ_π vanish,

$$\tilde{\kappa}_s(\sigma_s) = \frac{\sigma_s - \rho\sigma_x}{\rho\sigma_s - \sigma_x}\kappa_x$$

then it has a singularity at $sigma_s = \sigma_x/\rho$ where it changes the sign (which is only relevant for $\rho > 0$, since volatilities are nonnegative). The slope of this function

$$\frac{d\tilde{\kappa}_s}{d\sigma_s}(\sigma_s) = \frac{-(1-\rho^2)\sigma_x}{(\rho\sigma_s - \sigma_x)^2}\kappa_x$$

is everywhere nonpositive (since $\kappa_x \geq 0$). Thus if Condition (17) should not be satisfied independent of σ_s then κ_s must lie outside of the range of $\tilde{\kappa}_s(\sigma_s)$. Which gives Condition (18).

A.6 Proof of Proposition 4

If the dynamics of the cash flow and of the belief given in the proposition are the risk neutral dynamics (i.e., the dynamics under the equivalent martingale measure), then claim valuation can be done with respect to the expected value. Then the claim value is defined by

$$F(x,\pi) = E\left(\int_0^T (kx_t + d)e^{-rt}dt + \Phi(x,\pi,T)e^{-rT}\right)$$

where T is some random stopping time and Φ is a terminal condition. Using dynamic programming one finds that the value function must satisfy

$$F(x,\pi) = (kx_t + d)dt + e^{-r\,dt}E\left(F(x,\pi) + dF(x,\pi)t\right).$$

Applying Ito's Lemma and suppression of terms of the order of $o(dt)$ leads to the Valuation Equation (15) which confirms the proposition, that the given SDEs characterize the risk neutral dynamics of x and π.

A.7 Proof of Proposition 5

Assume that $\varphi(p,t)$ is the density of the belief at time t (when starting from a given prior). From the risk neutral dynamics of the Bayesian belief π_t (see Proposition 4) it follows that the boundaries are inaccessible, i.e., $\varphi(0,t) = \varphi(1,t) = 0$ for $t > 0$. This belief density has to satisfy the Fokker-Planck or Kolmogorov forward equation (see e.g., Merton (1990))

$$\begin{aligned}\frac{\partial\varphi(p,t)}{\partial t} &= \frac{1}{2}\frac{\partial^2}{\partial p^2}\left[\left(\kappa_\pi\frac{p(1-p)(\mu_h-\mu_l)}{\sigma_y}\right)^2\varphi(p,t)\right] \\ &\quad -\frac{\partial}{\partial p}\left[\left(-p\lambda_{hl} + (1-p)\lambda_{lh} - (\kappa_\pi)_t\frac{p(1-p)(\mu_h-\mu_l)}{\sigma_y}\right)\varphi(p,t)\right]\end{aligned}$$

Assume, the stationary density $\varphi(p)$ exists, then it must satisfy $\varphi(p) = \lim_{t\to\infty}\varphi(p,t)$, thus, existence requires $\lim_{t\to\infty}\frac{\partial\varphi(p,t)}{\partial t} = 0$, or in other words, if the stationary belief density exists, it is the unique solution to

$$\begin{aligned}0 &= \frac{1}{2}\frac{\partial^2}{\partial p^2}\left[\left(\kappa_\pi\frac{p(1-p)(\mu_h-\mu_l)}{\sigma_y}\right)^2\varphi(p)\right] \\ &\quad -\frac{\partial}{\partial p}\left[\left(-p\lambda_{hl} + (1-p)\lambda_{lh} - \kappa_\pi\frac{p(1-p)(\mu_h-\mu_l)}{\sigma_y}\right)\varphi(p)\right],\end{aligned}$$

$$\text{s.t.}\quad \begin{aligned}\textstyle\int_0^1\varphi(p)dp &= 1,\\ \varphi(0) = \varphi(1) &= 0.\end{aligned}$$

The expression in Proposition 5 satisfies this conditions, and is therefore the stationary belief density.

A.8 Proof of Proposition 6

Under the risk neutral measure the value of a claim is determined by the expected value of the integrated discounted cash flow. Being in the high growth state, the probability of staying in this state over a time interval of length dt is $1-\lambda_{hl}dt$, see Assumption 2. With probability $\lambda_{hl}dt$, the state changes. Therefore, using the theorem of Feynman and Kac F_0 and F_1 can be written as

$$\begin{aligned} F_0(x) &= kx+d+e^{-r\,dt}[(1-\lambda_{hl}dt)E(F_0+dF_0)+\lambda_{hl}dt(F_1(x)-F_0(x))], \\ F_1(x) &= kx+d+e^{-r\,dt}[(1-\lambda_{lh}dt)E(F_1+dF_1)+\lambda_{lh}dt(F_0(x)-F_1(x))]. \end{aligned}$$

Series expansion of dF_0 and dF_1 and using Ito's Lemma gives the system of differential equations stated in the proposition.

A.9 Proof of Proposition 7

The value of the perpetual flow is given by

$$X(x)=E\left(\int_0^\infty x_t e^{-r\tau}d\tau\right)\bigg|_{x_0=x}.$$

Since the upper integration boundary is ∞, the specification of the cash flow dynamics implies linearity of X in x, i.e.,

$$X(\xi x)=\xi X(x)$$

for any constant ξ. Thus, substituting linear functions $X_0(x)=\gamma_0 x$ and $X_1(x)=\gamma_1 x$ into the system (25) and recognizing that for the perpetual flow we have $k=1$ and $d=0$ yield the expressions for γ_0 and γ_1 which are stated in Equation (27) of the proposition.

A.10 Proof of Proposition 8

Equity can be interpreted as a claim on the perpetual flow minus the value of the tax adjusted perpetual coupon flow plus a put option that allows equityholders to step aside by defaulting on their obligations. Therefore, as long as equityholders run the firm, the value of equity has to satisfy the system (25) with $k=1$ and $d=-(1-\tau)c$. Since equityholders' decision to default is conditioned on the current growth state, one has to distinguish between three cases, (i) the level of the cash flow is such that equityholders maintain their obligation independent of the growth state, i.e., $x>\underline{x}_0$, (ii) the level of the cash flow is such that equityholders default in the low growth state whereas they run the firm if it is in the high growth state, i.e., $\underline{x}_1<x\leq\underline{x}_0$, and (iii) the firm is handed over to debtholders immediately, independent of the growth state, i.e., $x\leq\underline{x}_1$.

In case (iii) we have $E_0=E1=0$. In case (ii) it is clear that $E_0=0$, because equityholders default in the case of low growth. Since default is an unrevocable decision, the second equation in the system (25) becomes irrelevant. Substituting $E_0 = 0$ in the first equation of the system (25) gives a second order ordinary differential equation which is standard in option pricing

$$(r+\lambda_{hl}E_1(x) = \frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 E_1}{\partial x^2}(x) + \mu_h x \frac{\partial E_1}{\partial x}(x) + x - (1-\tau)c.$$

A particular solution of the equation is $\frac{x}{r+\lambda_{hl}-\mu_h} + \frac{c}{r+\lambda_{hl}}$. The homogeneous equation can be solved using the ansatz $E = x^\beta$. So the general solution is

$$E_{13}\, x^{\beta_1} + E_{14}\, x^{\beta_2} + \frac{x}{r+\lambda_{hl}-\mu_h} - \frac{(1-\tau)c}{r+\lambda_{hl}},$$

where E_{13} and E_{14} are constants to be determined by boundary conditions and β_1 and β_2 are the roots of the characteristic quadratic polynomial

$$R(\beta) = \frac{1}{2}\sigma_x^2\beta^2 + (\hat{\mu} - \frac{1}{2}\sigma_x^2)\beta - (r+\lambda_{hl}).$$

In case (i) a particular solution to the system (25) can be found by using the result of the previous proposition (i.e., Proposition 7),

$$\gamma_0 x - \frac{(1-\tau)c}{r} \quad \text{if } \mu_t = \mu_l, \gamma_1 x - \frac{(1-\tau)c}{r} \quad \text{if } \mu_t = \mu_h,$$

To obtain the general solution of the homogenous equations one can write E_2 as a function of E_1 and its first and second derivative

$$E_2(x) = \frac{1}{\lambda_{hl}}\left[rE_1(x) - \frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 E_1}{\partial x^2}(x) - \mu_h x \frac{\partial E_1}{\partial x}(x) + \lambda_{hl}E_1(x)\right]$$

Substituting this equation into the second differential equation of the system eliminates the E_2 and gives a fourth order ordinary differential equation involving E_1. The ansatz x^η solves the homogenous equation given that η is a root of the characteristic polynomial

$$\begin{aligned} Q(\eta) \quad = \quad & \sigma_x^4\eta^4 \\ & +2\sigma_x^2(\hat{\mu}_h + \hat{\mu}_l - \sigma_x^2)\eta^3 \\ & +[4\hat{\mu}_h\hat{\mu}_l - \sigma_x^2(2(\lambda_{hl} + \lambda_{lh} + \hat{\mu}_h + \hat{\mu}_l + 2r) - \sigma_x^2)]\eta^2 \\ & +[2\sigma_x^2(\lambda_{hl} + \lambda_{lh} + 2r) - 4(\lambda_{lh}\hat{\mu}_h + \lambda_{hl}\hat{\mu}_l + \hat{\mu}_h r + \hat{\mu}_l r)]\eta \\ & +4r(\lambda_{hl} + \lambda_{lh} + r) \end{aligned} \tag{43}$$

Under the conditions $\hat{\mu}_h < r$ and $\hat{\mu}_l < r$ (which are satisfied due to Assumption 8) this polynomial has two negative roots and two positive roots which exceed 1. The general solution of the homogenous equation is

$$E_1\,x^{\eta_1} + E_2\,x^{\eta_2} + E_3\,x^{\eta_3} + E_4\,x^{\eta_4},$$

where $\eta_1, \eta_2 < 0$, $\eta_3, \eta_4 > 1$ are the four roots of the characteristic polynomial and E_1, E_2, E_3, E_4 are constants to be determined by boundary conditions. The exclusion of speculative bubbles requires $E_3 = E_4 = 0$. Substitution of the solution for E_1 into the expression for E_2 leads to the expressions for Δ_1 and Δ_2. Being in the high growth state, the level of the cash flow can freely cross the boundary $\underline{x}_1$, thus, E_1 has to be continuous and differentiable at $\underline{x}_1$, which are two of the given boundary conditions. At the default thresholds, the value of equity vanishes, which determines the remaining two boundary conditions.

A.11 Proof of Proposition 9

The proof is analogous to the proof for the value of equity. The cash flow to debtholders is determined by $k = 0$ and $d = c$. In the case of default (which occurs exogenously from the point of view of debtholders) the value is given by a fraction ϕ of the value of the perpetual unlevered assets, this determines two of the boundary conditions. The argument for the remaining two conditions is the same as in the case of equity.

A.12 Proof of Proposition 10

The value X of the perpetual unlevered cash flow of the firm's assets must satisfy the second order partial differential Equation (15), see Proposition 3 with $k = 1$ and $d = 0$. Due to the theorem of Feynman and Kac this is equivalent to the integral equation

$$X(x,\pi) = E\left(\int_0^\infty x_t e^{-r\tau} d\tau\right)\bigg|_{x_0 = x, \pi_0 = \pi}.$$

Since the time horizon is infinite and due to the assumptions about the cash flow dynamics, X does not explicitly depend on time and it is linear in the level of the flow x, i.e., $X(\xi x, \pi) = \xi X(x,\pi)$. In the case of $\kappa_\pi = 0$ the dynamics of the belief π under the risk neutral measure are identical to the objective Bayesian learning dynamics. Therefore, an ansatz of the form $X(x,\pi) = x(\pi X(1,1) + (1-\pi)X(1,0))$ decomposes the partial differential equation (15) into a system of two ordinary differential equations which is identical to (25). Consequently, we get

$$\begin{aligned} X(1,1) &= \gamma_1 \\ X(1,0) &= \gamma_0 \end{aligned}$$

with γ_0, γ_1 given in Proposition 7.

A.13 Proof of Proposition 11

The value of equity has to satisfy Equation (15) with $k = 1$ and $d = -(1-\tau)c$. Analogous to the proof of Proposition 10 an ansatz of the form $E(x,\pi) = \pi E(x,1) + (1-\pi)E(x,0)$ decomposes the partial differential equation into a system of two ordinary differential equations identical to (25). Since exogenous default at a certain lower level $\underline{x}$ gives a boundary condition which preserves the linearity in the belief π, the solutions for $E(x,1)$ and $E(x,0)$ are the same as under perfect information in the case of $\underline{x}_1 = \underline{x}_0 = \underline{x}$.

A.14 Proof of Proposition 12

The proof is analogous to the proof for the value of equity with $k = 0$ and $d = c$.

A.15 Proof of Corollary 3

This corollary follows directly form the paper of Nelson and Ramaswamy (1990).

References

Tomas Björk. *Arbitrage Theory in Continuous Time*. Oxford University Press, New York, 1998.

Phelim P. Boyle, Jeremy Evnine, and Steven Gibbs. Numerical evaluation of multivariate contingent claims. *Review of Financial Studies*, 2(2):241–250, 1989.

Peter Ove Christensen, Christian Riis Flor, David Lando, and Kristian R. Miltersen. Dynamic capital structure with callable debt and debt renegotiations. workingpaper, 2000.

Thomas Dangl and Josef Zechner. Credit risk and dynamic capital structure choice. working paper, 2003.

Alexander David and Pietro Veronesi. Option prices with uncertain fundamentals. working paper, 2002.

Edwin O. Fischer, Robert Heinkel, and Josef Zechner. Dynamic capital structure choice: Theory and tests. *Journal of Finance*, 44:19–40, 1989.

Robert Goldstein, Nengjiu Ju, and Hayne Leland. An ebit-based model of dynamic capital structure. *Journal of Business*, 74(4):483–512, 2001.

Hayne E. Leland. Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance*, 49(4):1213–1252, 1994.

R. S. Liptser and A. N. Shiryaev. *Statistics of Random Processes I, General Theory*. Springer, 2000.

Pierre Mella-Barral. The dynamics of default and debt reorganization. *Review of Financial Studies*, 12 (3):535–578, 1999.

Pierre Mella-Barral and William Perraudin. Strategic debt service. *Journal of Finance*, 52(2):531–556, 1997.

Robert C. Merton. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, 29:449–469, 1974.

Robert C. Merton. *Continuous-time Finance*. Blackwell, Cambridge, MA, 1990.

Daniel B. Nelson and Krishna Ramaswamy. Simple binomial processes as diffusion approximations in financial models. *Review of Financial Studies*, 3(3):393–430, 1990.

Pietro Veronesi. Stock market overreaction on bad news in good times: A rational expectations equilibrium model. *The Review of Financial Studies*, 12(5):975–1007, 1999.

Pietro Veronesi. How does information quality affect stock returns? *The Journal of Finance*, 55(2): 807–837, 2000.