Dilution, Anti-Dilution, and Corporate Positions in Options on the Company's Own Stocks

M Hanke† and K Pötzelberger‡

†Department of Operations Research, Vienna University of Economics and Business Administration, Augasse 2-6, 1090 Vienna, Austria
‡Department of Statistics, Vienna University of Economics and Business Administration, Augasse 2-6, 1090 Vienna, Austria

E-mail: `Michael.Hanke@wu-wien.ac.at`, `Klaus.Poetzelberger@wu-wien.ac.at`

Abstract. In this paper, we analyze options that are bought or sold by the same company on whose stocks these options are written, leading to dilution and anti-dilution effects. We provide valuation equations for the European versions of such options, and discuss conditions for existence and uniqueness of their prices. Option prices to be paid or received for these options by the company are shown to be different from those that apply for standard options (which are bought and sold by outside investors). Since the options become part of the company's assets/liabilities, the stochastic process followed by the stock price changes. We demonstrate how the new stock price process can be derived, and discuss economic implications of our results. Numerical examples illustrate our findings.

1. Introduction

A derivative is usually defined as a security whose payoff depends on the value of some other ("underlying", "more basic") security on a given date or during a given period. In the vast majority of papers on option pricing, this definition is understood to imply that the price of the underlying has an impact on the price of the option, but not vice versa. This type of relation between an underlying and the options written on it goes without saying for people working in the field of derivatives. In fact, it seems so self-evident that it is hardly reflected any further. This is due to the fact that almost all practical situations are covered by the standard definition.

When, however, a company takes positions in options written on this company's stocks, the value of these option positions becomes part of the company's assets (in case of long positions) or liabilities (for short positions). Therefore, any move in the option price affects the market value of the company and thus the stock price. To distinguish this special situation from the standard case (where options are bought or sold by third parties), we propose the term OOO option (pronounced "triple-o option", short for "on-one's-own (stocks) option") to denote the situation described. This feedback effect between stock and option prices is different from the feedback effect caused by dynamic hedging, which has been discussed by Frey and Stremme (1997), Platen and Schweizer (1998), and Heath et al. (2001).

Surprisingly, in the option pricing literature, the general problem of pricing options bought or sold by the same company on whose stock these options are written has (to the best of our knowledge) not received any attention so far. Only the special case of warrants and the associated dilution effect has been dealt with extensively (see, e.g., Galai and Schneller (1978)). Murphy (1998) recommends buying OOO put options as a way of insuring against bankruptcy. However, even in papers focusing on the valuation of executive stock options (that are another special case of OOO options), the consequences of the fact that these options are written by companies on "themselves" are often ignored (for a recent publication in this field, see e.g. Carpenter (1998) and the references given there). The clear need for a general analysis of such transactions arises from the fact that they do occur in practice. In this paper, we fill this gap and provide valuation equations for the situation just described. We discuss the resulting dilution and anti-dilution effects and show how the stochastic process followed by the underlying changes as a result of such transactions.

The paper is organized as follows: In Section 2, we set up the framework for our analysis and show how to value and hedge European OOO call and put options. Then, we extend our analysis to more general (European-style) payoff functions and show that warrants can be regarded as a special case of our general problem. Section 3 provides a discussion of the existence and uniqueness of prices, together with basic inequalities for the prices of OOO options. Section 4 analyzes the effects of corporate positions in OOO options on the stochastic process followed by the stock price. In Section 5, we illustrate our results using numerical examples. Section 6 discusses practical implications of our results. A summary concludes, sketching related research questions currently investigated.

2. The Valuation of Cash-Settled European OOO Options

A *OOO option* is defined here as an option that is bought or sold by the same company on whose stocks it is written. For simplicity of exposition, we analyze a company that is initially fully equity-financed. The only security issued by this company is common stock. Throughout the paper, we analyze OOO options as fractions of options on the market value of the company (i.e., the stock price multiplied by the number of stocks outstanding, as opposed to options on individual stocks). This simplifies the notation considerably. We confine ourselves to European-style OOO options. For ease of exposition, we assume that the company pays no dividends. This assumption could easily be relaxed.

In this paper, we confine ourselves to OOO options with cash settlement. I.e., the parties to the transaction agree in advance that no shares will be exchanged upon exercise, but the party in the short position pays the party in the long position the difference between the stock price at maturity and the strike (for call options) or the negative of this difference (for put options). In this case, the simple exercise rule for OOO options is to exercise them if and only if they are in the money at maturity. In case of physical delivery, exercise rules become more complicated, since ownership and legal issues have to be taken into account (in many legislations, there are restrictions on the volume of corporate holdings in a company's own stocks). A practical example using the same exercise mode as in this paper are stock appreciation rights given to executives as part of their compensation packages.

Throughout the paper, we will use the following notation:
$X_t \ldots$market value of the company at time t,
$S_t \ldots$market value of the company's assets (excluding any effects from transactions in options as analyzed here) at time t,
$C_t(K, T, S_t) \ldots$time t price of a cash-settled standard European call option on S with strike price K and expiration time T (according to any specific pricing model, e.g., Black-Scholes),
$\gamma_t^{\{b,s\}}(K, T, S_t, p) \ldots$time t price of p cash-settled European OOO call options on X with strike price K and expiration time T, which are bought (γ_t^b) or sold (γ_t^s) by the same company on whose stock they are written,
$P_t(K, T, S_t) \ldots$time t price of a cash-settled standard European put option on S with strike price K and expiration time T (according to any specific pricing model, e.g., Black-Scholes),
$\pi_t^{\{b,s\}}(K, T, S_t, p) \ldots$time t price of p cash-settled European OOO put options on X with strike price K and expiration time T, which are bought (π_t^b) or sold (π_t^s) by the same company on whose stock they are written,
$B_{t,T} \ldots$discount factor applicable to period t, T, $B_{T,t} = 1/B_{t,T}$. For simplicity of

notation, we assume that interest rates are deterministic functions of time (cf. the remark in Section 2.1 for the stochastic case).

We use the following basic assumptions:

A1: Prices in our market model are calculated under an equivalent martingale measure. This includes the case where the equivalent martingale measure is unique as well as the case where the market is incomplete, but one specific martingale measure is used to calculate prices for some reason. To calculate the prices of OOO options, we use the same equivalent martingale measure. This implies, as will become clear in the analysis, that market participants are informed about the terms of the company's transactions in OOO options. Otherwise, arbitrage opportunities would arise.

A2: There exists a stochastic process (S_t) describing the market value of the company's assets, adapted to a filtration $\mathcal{F}_t$. We note that A1 implies certain regularity assumptions for (S_t), such that prices can be calculated as expectations under an equivalent martingale measure.

A3: The transactions in OOO options do not carry any information beyond the terms of the transactions themselves. In other words, there is no information asymmetry, and there are no signaling effects associated with these transactions. The market neither reacts positively nor negatively to the information that a company buys or sells OOO options.

The economic background we have in mind is a company that approaches a financial institution and wants to arrange the corresponding option deal with this bank. In this situation, all parties to the transaction are informed about the fact that it is the company itself who buys or sells these options, which is important for valuation purposes.

In the next subsection, we start with European OOO call options sold by the company (this is largely analogous to the valuation of warrants, which is the only special case of OOO options which has received considerable attention in the literature). In the following subsections, we state the results for the remaining three basic positions (short and long put, long call) together with differences in the derivations.

2.1. Cash-settled European OOO Calls Sold by the Company

We assume that at some time t_0, the company sells a fraction p of OOO call options on its market value X with exercise price K and expiration time T for a price of $\gamma^b_{t_0}(K, T, S_{t_0}, p) =: \gamma$. Therefore, the process of the company's market value reads as

$$X_t = \begin{cases} S_t & \text{for } 0 \leq t < t_0 \\ S_t + B_{t,t_0}\gamma - V_t & \text{for } t_0 \leq t \leq T \end{cases},$$

where we have dropped the dependencies for notational convenience. Here and in the following subsections, we assume that the company borrows the money needed to pay (or invests the money received, resp.) for the options at the risk-free interest rate. V_t denotes the value of the options at time t. Both γ and V_t are yet to be determined. To specify γ and V_t, we note that at maturity of the OOO options, the value of the company is

$$X_T = S_T + B_{T,t_0}\gamma - p(X_T - K)_+.$$

I.e., if $X_T \leq K$,

$$X_T = S_T + B_{T,t_0}\gamma. \tag{1}$$

In this case, the value of the company is too low for the calls to be exercised. If, however, $X_T > K$ (the calls are in the money), the calls will be exercised and some algebra leads to

$$X_T = \frac{S_T + B_{T,t_0}\gamma + pK}{1+p}. \tag{2}$$

Equating (1) and (2) shows that, in terms of S_T, switching between (1) and (2) occurs at

$$S_T = K - B_{T,t_0}\gamma.$$

I.e., if $S_T \leq K - B_{T,t_0}\gamma$,

$$X_T = S_T + B_{T,t_0}\gamma.$$

If $S_T > K - B_{T,t_0}\gamma$,

$$X_T = \frac{S_T + B_{T,t_0}\gamma + pK}{1+p}.$$

The value of p OOO call options on X_T at maturity is thus given by

$$p(X_T - K)_+ = \begin{cases} p(S_T + B_{T,t_0}\gamma - K)_+ & \text{for } S_T \leq K - B_{T,t_0}\gamma \\ p(\dfrac{S_T + B_{T,t_0}\gamma + pK}{1+p} - K)_+ & \text{for } S_T > K - B_{T,t_0}\gamma \end{cases}. \tag{3}$$

The first line in (3) corresponds to the case when the calls expire worthless. Therefore, (3) can be simplified to

$$p(X_T - K)_+ = \frac{p}{1+p}(S_T + B_{T,t_0}\gamma - K)_+. \tag{4}$$

This is equivalent to the value of $p(1+p)^{-1}$ standard European call options on S_T (as opposed to X_T!) with strike price $K - B_{T,t_0}\gamma$.

To derive the correct price γ of the OOO call options sold by the company, we invoke an arbitrage argument: The company, as the seller of the p OOO call options, has received γ (at time $t = t_0$) in exchange for the obligation to pay $p(X_T - K)_+$ (at time $t = T$). As shown in (4), it can hedge by duplicating $p(1+p)^{-1}$ standard European call options on S_T with strike $K - B_{T,t_0}\gamma$. This is possible because S_t, though not directly observable, can be calculated from X_t. To rule out arbitrage opportunities, the price γ of the p OOO calls must satisfy

$$\gamma = \frac{p}{1+p} C_{t_0}(K - B_{T,t_0}\gamma, T, S_{t_0}). \tag{5}$$

If S_t follows a geometric Brownian motion, $C(\cdot)$ is the price of a call option according to the Black-Scholes formula. The valuation equation (5) can be solved numerically for γ. In Section 3, we show that the correct price γ of OOO call options is always smaller than the price of p standard call options (that are not sold by the company, but by outside investors). Proofs for existence and uniqueness of the solution are also given there together with basic inequalities for the values of the various different option positions described in this section.

The factor $p(1+p)^{-1}$ represents the well-known *dilution effect* associated with call options sold by the company, which are usually called warrants if coupled with issuance of additional stocks in case of exercise of the options. Dilution means that the (original) shareholders sustain a loss if the options are exercised. Neither the existence, nor the extent of this effect depend on the exercise mode (physical delivery of newly issued shares vs. cash settlement).

Remark. In case of stochastic interest rates, (5) reads as

$$\gamma = \mathbb{E}^Q \left[B_{T,t_0} \frac{p}{1+p} (S_T + B_{T,t_0}\gamma - K)_+ | \mathcal{F}_{t_0} \right], \tag{6}$$

where Q denotes the martingale measure. Thus, we have to hedge a standard European call with strike price K on the underlying $S_t - B_{t,t_0}\gamma$.

2.2. Cash-settled European OOO Puts Bought by the Company

In the case of OOO put options bought by the company, the process for X_t becomes

$$X_t = \begin{cases} S_t & \text{for } 0 \le t < t_0 \\ S_t - B_{t,t_0}\pi + V_t & \text{for } t_0 \le t \le T \end{cases}.$$

Following the steps described above leads to the following valuation equation for OOO put options bought by the company:

$$\pi = \frac{p}{1+p} P_{t_0}(K + B_{T,t_0}\pi, T, S_{t_0}).$$

The value π of OOO put options bought by the company is always smaller than or equal to‡ the value of otherwise similar put options bought by outside investors (cf. Section 3).

2.3. Cash-settled European OOO Calls Bought by the Company

Denoting by p the fraction of OOO call options on the company's market value X bought by the company in t_0, the process for X_t becomes

$$X_t = \begin{cases} S_t & \text{for } 0 \le t < t_0 \\ S_t - B_{t,t_0}\gamma + V_t & \text{for } t_0 \le t \le T \end{cases}. \tag{7}$$

An analysis following the same lines as above leads to the following valuation equation:

$$\gamma = \frac{p}{1-p} C_{t_0}(K + B_{T,t_0}\gamma, T, S_{t_0}).$$

‡ For $\mathrm{P}(S_T > K) > 0$, the inequality is strict.

Thus, the value of the OOO calls is undefined for $p = 1$. In fact, Section 3 shows that existence and uniqueness of the OOO call price is only guaranteed for $0 \le p < 1$. Furthermore, the value γ of OOO calls bought by the company (if it exists) is always larger than the value of otherwise similar call options bought by outside investors.

2.4. Cash-settled European OOO Puts Sold by the Company

In the case of OOO put options sold by the company, the process for the company's market value X_t reads as

$$X_t = \begin{cases} S_t & 0 \le t < t_0, \\ S_t + B_{t,t_0}\pi - V_t & t_0 \le t \le T. \end{cases}$$

The corresponding valuation equation becomes

$$\pi = \frac{p}{1-p} P_{t_0}(K - B_{T,t_0}\pi, T, S_{t_0}). \tag{8}$$

Similar to the case of long OOO calls discussed above, the value of the short OOO put is undefined for $p = 1$, and existence and uniqueness are only guaranteed for $0 \le p < 1$. The value π of OOO puts sold by the company (if it exists!) is always larger than the value of otherwise similar put options sold by outside investors (cf. Section 3).

Remark. When OOO puts are sold or OOO calls are bought, X_t as defined in the corresponding subsections may become negative (for positive values of S_t!), which is in contrast to the limited liability of shareholders. The two cases, however, are structurally different.

When OOO calls are bought and X_t reaches zero, the investment of the company into call options simply leads to the ruin of the company. Initially, the life of the company would have ended if the value of the productive assets, S_t, had fallen to zero. Now, the company goes bankrupt if S_t falls below $B_{T,t_0}\gamma$, the price paid for the calls plus accrued interest (cf. (7)).

When OOO puts are sold, the effect of X_T becoming zero (or negative) is the default of the company on the put obligation. Therefore, to avoid additional complications such as an analysis of the risk of default (which are non-central to this paper), we assume that X_t is always positive. A sufficient condition for this is that $\mathrm{P}(S_t \ge pK) = 1$, i.e., the quantity p of OOO put options sold is restricted, and the value of the productive assets is bounded below by a suitable positive constant.

2.5. Effects on the Market Value of the Company at Maturity of the Option

Figure 1 shows the effects of the four different positions in cash-settled OOO options analyzed above on the market value of the company at maturity. In each case, the thin line indicates the value the company would have had at time T without any transactions in options. Then, X_T would simply be equal to S_T, the market value of the company's assets. The bold line in each graph indicates the time T value of the company given the transaction in OOO options at time t_0. The slope of the bold line in the region where exercise is profitable depends on the fraction of options purchased: The more options bought or sold, the larger the slope's (absolute) deviation from 1. In the case of short

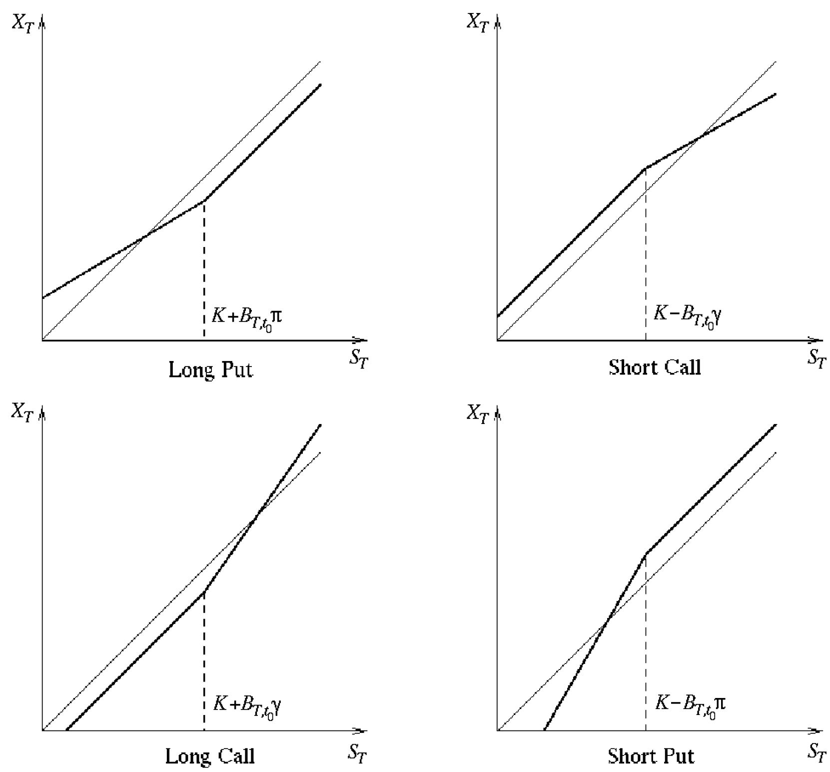

Figure 1. Market value X_T of the company at expiration time, depending on the market value of the productive assets S_T and the position in OOO options taken by the company

positions, the dilution effect discussed in Section 2.1 occurs: In case of exercise of the options, shareholders lose. For long positions, the opposite is true: Shareholders gain if the options are exercised. In the absence of any existing terminology, we will use the term *anti-dilution* to denote this effect.

Since X_T changes as a result of the option transaction, arbitrage considerations imply that the process (X_t) is affected also for $t < T$ when the company takes positions in options written on its own stocks. These changes are analyzed in Section 4.

The value of X_T if p OOO call options have been bought at $t < T$ is different from the value X_T would have if p OOO put options had been bought. Therefore, standard put-call parity obviously does not hold for OOO options.

2.6. General Solution for Cash-Settled European-Style OOO Options

In this subsection, we derive a valuation equation for more general European-style payoff functions. Denoting the payoff of the OOO option under consideration by $h(X_T)$, we note that the company's market value at maturity of the option can be written as§

$$X_T = S_T - B_{T,t_0}\kappa + ph(X_T) \tag{9}$$

when the option under consideration is held in a long position (otherwise, the signs of the last two terms would be reversed). Here, κ denotes the price paid for p options. I.e., (if (9) is invertible),

$$X_T = f(S_T).$$

According to standard asset pricing theory, the value V_t of the p options can be written as

$$V_t = p\mathbb{E}^Q\left[B_{t,T}h(X_T)|\mathcal{F}_t\right].$$

Here, Q denotes the equivalent martingale measure. Then, we find that

$$X_t = S_t - B_{t,t_0}\kappa + p\mathbb{E}^Q\left[B_{t,T}h(X_T)|\mathcal{F}_t\right]$$

and further

$$X_t = S_t - B_{t,t_0}\kappa + p\mathbb{E}^Q\left[B_{t,T}h(f(S_T))|\mathcal{F}_t\right]. \tag{10}$$

Again, in case of a short option position, the signs of $B_{t,t_0}\kappa$ and the expectation term would be reversed. For the possible problem of the company defaulting on the put obligation, cf. the corresponding remark in Section 2.4.

A related problem is the valuation of European warrants issued by a company. A warrant is a OOO call option where, upon exercise, new stocks are issued by the company and delivered to the warrant holder. In this case, (X_t) reads as

$$X_t = \begin{cases} S_t & \text{for } 0 \le t < t_0 \\ S_t + B_{T,t_0}\kappa + \mathbb{E}^Q\left[B_{t,T}h(f(S_T))|\mathcal{F}_t\right] & \text{for } t_0 \le t \le T \end{cases},$$

with $h(f(S_T)) = KI_{\{q(S_T+B_{T,t_0}\kappa+K)>K\}}$. Here, $q = m/(n+m)$, where n is the number of old stocks and m is the number of new stocks that are issued in case the warrants are exercised. The derivation of the valuation equation for warrants in our framework together with subsequent changes in the stock price process and the resulting changes in prices of traded options is shown in detail in Hanke/Pötzelberger (2002).

2.7. Example: Several Cash-Settled OOO Puts Bought by a Company

In this section, we extend an idea originally put forward by Murphy (1998) who recommends buying OOO put options to insure against bankruptcy. We use several OOO puts with different strike prices to generate a "multi-level bankruptcy insurance", applying the valuation principles derived in the previous subsection.

Here, we still assume that initially, our company is fully equity-financed. At time t_0, it buys fractions p_n of n put options with different strikes K_n, but the same expiration time T on its value X. Using the notation introduced in the previous subsection,

$$h(X_T) = p_1(K_1 - X_T)_+ + p_2(K_2 - X_T)_+ + \cdots + p_n(K_n - X_T)_+,$$

§ Here, for the sake of continuity, we keep up with the notion of buying a fraction p of an option with a payoff of $h(X_T)$.

where $0 \leq K_1 < K_2 < \ldots < K_n < \infty$. Denoting by I the indicator function, this terminal payoff can be rewritten as ($K_0 = 0$)

$$\begin{aligned} h(X_T) &= \sum_{i=1}^{n} p_i (K_i - X_T)_+ \\ &= \sum_{i=1}^{n} I_{[K_{i-1},K_i[}(X_T) \left(\sum_{j=i}^{n} p_j (K_j - X_T) \right) \\ &= \sum_{i=1}^{n} I_{[K_{i-1},K_i[}(X_T) \left(\sum_{j=i}^{n} p_j K_j - \sum_{j=i}^{n} p_j X_T \right). \end{aligned} \tag{11}$$

We denote by π the total price paid by the company for this package of put options. Recalling from (9) that

$$X_T = S_T - B_{T,t_0}\pi + h(X_T)$$

and using (11), we can rewrite the market value of the company at maturity as

$$X_T = S_T - B_{T,t_0}\pi + \sum_{i=1}^{n} I_{[K_{i-1},K_i[}(X_T) \left(\sum_{j=i}^{n} p_j K_j - \sum_{j=i}^{n} p_j X_T \right). \tag{12}$$

It is easily seen from (12) that if X_T is larger than K_n (i.e., none of the puts is exercised), $X_T = S_T - B_{T,t_0}\pi$. This means that the value of the company is just the value of the productive assets minus the premium paid for the puts. For X_T in a specific interval $[K_{i-1}, K_i[$, we see from (12) that

$$X_T = \frac{S_T - B_{T,t_0}\pi + \sum_{j=i}^{n} p_j K_j}{1 + \sum_{j=i}^{n} p_j} \quad \text{for } X_T \in [K_{i-1}, K_i[. \tag{13}$$

Therefore, X_T can be rewritten as

$$X_T = \sum_{i=1}^{n} I_{[K_{i-1},K_i[}(X_T) \frac{S_T - B_{T,t_0}\pi + \sum_{j=i}^{n} p_j K_j}{1 + \sum_{j=i}^{n} p_j} + I_{[K_n,\infty[}(X_T) \left(S_T - B_{T,t_0}\pi \right).$$

Redirecting the focus to X_T from a specific interval $[K_{i-1}, K_i[$ and using (13), we find that

$$K_{i-1} \leq \frac{S_T - B_{T,t_0}\pi + \sum_{j=i}^{n} p_j K_j}{1 + \sum_{j=i}^{n} p_j} < K_i$$

and further

$$K_{i-1} \left(1 + \sum_{j=i}^{n} p_j \right) - \sum_{j=i}^{n} p_j K_j + B_{T,t_0}\pi \leq S_T < K_i \left(1 + \sum_{j=i}^{n} p_j \right) - \sum_{j=i}^{n} p_j K_j + B_{T,t_0}\pi, \tag{14}$$

$$K_{i-1} + B_{T,t_0}\pi + \sum_{j=i}^{n} p_j \left(K_{i-1} - K_j \right) \leq S_T < K_i + B_{T,t_0}\pi + \sum_{j=i}^{n} p_j \left(K_i - K_j \right). \tag{15}$$

Denoting the left-hand part in (14) by u_i and the right-hand part in (15) by v_i, we find that $u_i = v_{i-1}$. Thus, there is a one-to-one correspondence between the boundaries K_i

of the intervals for X_T and the boundaries v_i of the intervals for S_T. Therefore, X_T can be rewritten as

$$\begin{aligned} X_T &= \sum_{i=1}^{n} I_{[v_{i-1},v_i[}(S_T) \frac{S_T - B_{T,t_0}\pi + \sum_{j=i}^{n} p_j K_j}{1 + \sum_{j=i}^{n} p_j} + I_{[v_n,\infty[}(S_T)\left(S_T - B_{T,t_0}\pi\right) \\ &= \left(S_T - B_{T,t_0}\pi\right) + \sum_{i=1}^{n} I_{[0,v_i[}(S_T)\left(\frac{v_i - S_T}{\beta_i}\right) \\ &= S_T - B_{T,t_0}\pi + \sum_{i=1}^{n} \frac{1}{\beta_i}\left(v_i - S_T\right)_+ . \end{aligned} \tag{16}$$

The valuation equation, when formulated as in (16), leads to a nice geometric interpretation. To see this, we first find an expression for β_i by noting that for a specific interval $[v_{i-1}, v_i]$,

$$\frac{S_T - B_{T,t_0}\pi + \sum_{j=i}^{n} p_j K_j}{1 + \sum_{j=i}^{n} p_j} = S_T - B_{T,t_0}\pi + \sum_{j=1}^{i} \frac{v_j - S_T}{\beta_j},$$

$$\frac{1}{1 + \sum_{j=i}^{n} p_j} = 1 - \sum_{j=1}^{i} \frac{1}{\beta_j},$$

$$\frac{1}{\beta_i} = \frac{1}{1 + \sum_{j=i}^{n} p_j} - \frac{1}{1 + \sum_{j=i-1}^{n} p_j},$$

$$\beta_i = \frac{\left(1 + \sum_{j=i}^{n} p_j\right)\left(1 + \sum_{j=i-1}^{n} p_j\right)}{p_{i-1}}.$$

Figure 2 shows that the slope of X_T is equal to 1 if X_T is large enough for none of the puts to expire in the money. Every put that expires in the money decreases the slope of X_T, thus providing more and more "insurance" against the company's market value falling to the same extent as the market value of the company's productive assets.

3. Existence of Prices and Basic Inequalities

Let a process (S_t) be given. A process (X_t) satisfying (9) does not necessarily exist, even if $X_t \mapsto X_t - B_{t,t_0}\kappa + ph(X_t)$ is invertible. If such a process exists, the inverse f_κ of $X_t \mapsto X_t - B_{t,t_0}\kappa + ph(X_t)$ depends on κ and κ satisfies

$$\kappa = \mathbb{E}^Q\left[B_{t_0,T}\phi_\kappa(S_T) \mid \mathcal{F}_{t_0}\right],$$

where $\phi_\kappa = p(h \circ f_\kappa)$ and κ is nonnegative. The following four propositions summarize basic facts about put or call options that are bought or sold. We assume $0 < B_{t_0,T} \leq 1$.

Proposition 3.1. *If p cash-settled OOO put options on its market value are* bought *by the company, a unique price exists. Denote by $\pi^b_{t_0}(K, T, S_{t_0}, p)$ the price of p options bought. Then*

$$\frac{p}{1+p} P_{t_0}(K, T, S_{t_0}) \leq \pi^b_{t_0}(K, T, S_{t_0}, p) \leq p P_{t_0}(K, T, S_{t_0}), \tag{17}$$

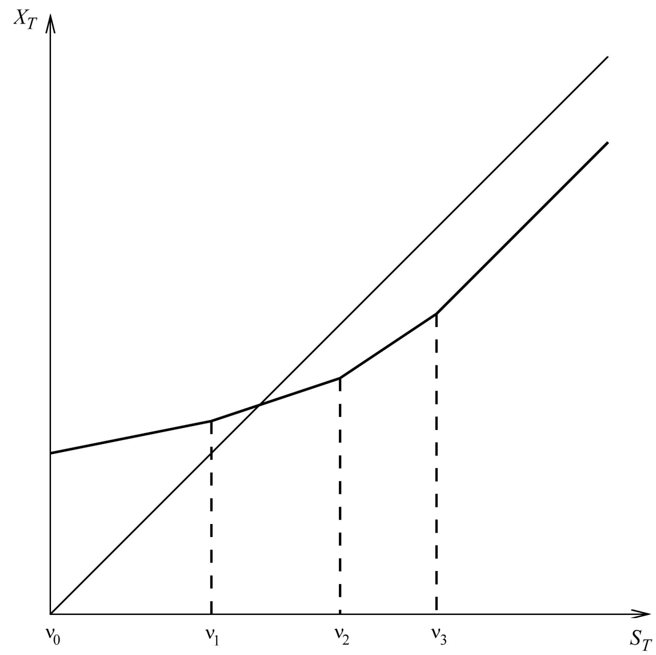

Figure 2. Market value X_T of the company at expiration time after buying several OOO put options on its value with different strike prices

and

$$p(B_{t_0,T}K - S_{t_0})_+ \leq \pi^b_{t_0}(K, T, S_{t_0}, p). \tag{18}$$

Proof. A positive π is sought with

$$\pi = \frac{p}{1+p}\mathbb{E}^Q \left[B_{t_0,T}(K + B_{T,t_0}\pi - S_T)_+ \mid \mathcal{F}_{t_0}\right]. \tag{19}$$

Denote the right-hand side of (19) by $\alpha(\pi)$. $\alpha(\pi)$ is nondecreasing with $\alpha(0) \geq 0$. Suppose, $\alpha(\pi) > \pi$ for all $\pi \geq 0$. Divide (19) by π and take the limit for $\pi \to \infty$. Then

$$1 \leq \frac{p}{1+p} < 1,$$

a contradiction. Thus at least one π solving (19) exists. Suppose there are two such, π_1 and π_2 with $\pi_1 < \pi_2$.

$$(K + B_{T,t_0}\pi_2 - S_T)_+ - (K + B_{T,t_0}\pi_1 - S_T)_+ \leq (\pi_2 - \pi_1)B_{T,t_0}$$

implies

$$\begin{aligned}\pi_2 - \pi_1 &= \frac{p}{1+p}\mathbb{E}^Q \left[B_{t_0,T}\left((K + B_{T,t_0}\pi_2 - S_T)_+ - (K + B_{T,t_0}\pi_1 - S_T)_+\right) \mid \mathcal{F}_{t_0}\right] \\ &\leq \frac{p}{1+p}(\pi_2 - \pi_1),\end{aligned}$$

a contradiction. To prove (17), note that the left-hand inequality follows from (19). Furthermore, $(K + B_{T,t_0}\pi - S_T)_+ \leq (K - S_T)_+ + B_{T,t_0}\pi$, so that, using (19),

$$\pi \leq \frac{p}{1+p}\mathbb{E}^Q\left[B_{t_0,T}\left((K - S_T)_+ + B_{T,t_0}\pi\right) \mid \mathcal{F}_{t_0}\right]$$
$$= \frac{p}{1+p}\left(P_{t_0}(K,T,S_{t_0}) + \pi\right),$$

which implies

$$\pi^b_{t_0}(K,T,S_{t_0},p) \leq pP_{t_0}(K,T,S_{t_0}).$$

$(K + B_{T,t_0}\pi - S_T)_+ \geq K + B_{T,t_0}\pi - S_T$ leads to

$$p(B_{t_0,T}K - S_{t_0}) \leq \pi^b_{t_0}(K,T,S_{t_0},p).$$

(18) follows, since $\pi^b_{t_0}(K,T,S_{t_0},p)$ is nonnegative. □

Proposition 3.2. *Let p cash-settled OOO call options on its market value be* sold *by the company. These options have a unique price $\gamma^s_{t_0}(K,T,S_{t_0},p)$. If $S_{t_0} \leq B_{t_0,T}K(1+1/p)$, then $\gamma^s_{t_0}(K,T,S_{t_0},p) \leq KB_{t_0,T}$. If $S_{t_0} > B_{t_0,T}K(1+1/p)$, then $\gamma^s_{t_0}(K,T,S_{t_0},p) = p(S_{t_0} - B_{t_0,T}K)$. Furthermore,*

$$\frac{p}{1+p}C_{t_0}(K,T,S_{t_0}) \leq \gamma^s_{t_0}(K,T,S_{t_0},p) \leq pC_{t_0}(K,T,S_{t_0}). \tag{20}$$

and

$$p(S_{t_0} - \mathbb{E}^Q[B_{t_0,T}|\mathcal{F}_{t_0}]K)_+ \leq \gamma^s_{t_0}(K,T,S_{t_0},p). \tag{21}$$

Proof. We look for a solution γ of

$$\gamma = \frac{p}{1+p}\mathbb{E}^Q\left[B_{t_0,T}(S_T - K + B_{T,t_0}\gamma)_+ \mid \mathcal{F}_{t_0}\right]. \tag{22}$$

Suppose, the right-hand side of (22) is always larger than γ for all $\gamma \in [0, B_{t_0,T}K]$. Then

$$B_{t_0,T}K < \frac{p}{1+p}\mathbb{E}^Q\left[B_{t_0,T}(S_T - K + K)_+ \mid \mathcal{F}_{t_0}\right] = \frac{p}{1+p}S_{t_0},$$

so that $S_{t_0} > B_{t_0,T}K(1+1/p)$. Moreover, (22) has a solution in $[0, B_{t_0,T}K]$ if $S_{t_0} \leq B_{t_0,T}K(1+1/p)$. Now, if $S_{t_0} > B_{t_0,T}K(1+1/p)$, define $\hat{\gamma} = p(S_{t_0} - B_{t_0,T}K)$. Then

$$\hat{\gamma} \geq p\left(B_{t_0,T}K(1+\frac{1}{p}) - B_{t_0,T}K\right) = B_{t_0,T}K,$$

so that

$$\frac{p}{1+p}\mathbb{E}^Q\left[B_{t_0,T}(S_T - K + B_{T,t_0}\hat{\gamma})_+ \mid \mathcal{F}_{t_0}\right] = \frac{p}{1+p}\mathbb{E}^Q\left[B_{t_0,T}(S_T - K + B_{T,t_0}\hat{\gamma}) \mid \mathcal{F}_{t_0}\right]$$
$$= \frac{p}{1+p}(S_{t_0} + \hat{\gamma} - B_{t_0,T}K) = \hat{\gamma}.$$

(20) and (21) are proved similarly to the corresponding statements in Proposition 3.1. □

Proposition 3.3. *Let p cash-settled OOO put options on its market value be* sold *by the company. These options have a unique price $\pi^s_{t_0}(K,T,S_{t_0},p)$ if $0 \leq p < 1$ and $P(S_T \geq pK) = 1$ (cf. the corresponding remark in Section 2). Furthermore,*

$$P_{t_0}(K,T,S_{t_0}) \leq \pi^s_{t_0}(K,T,S_{t_0},p) \leq \frac{p}{1-p} P_{t_0}(K,T,S_{t_0}), \tag{23}$$

and

$$p(B_{t_0,T}K - S_{t_0})_+ \leq \pi^s_{t_0}(K,T,S_{t_0},p). \tag{24}$$

Proof. The price $\pi = \pi^s_{t_0}(K,T,S_{t_0})$ is a solution of (8). Note that the right-hand side of (8) is decreasing in π and bounded by $p(1-p)^{-1}P_{t_0}(K,T,S_{t_0})$, which implies the existence of a unique solution. (23) and (24) are proved similarly to (17) and (18). □

Proposition 3.4. *Let p cash-settled OOO call options on its market value be* bought *by the company. These options have a unique price $\gamma^b_{t_0}(K,T,S_{t_0},p)$ if $0 \leq p < 1$. Furthermore,*

$$pC_{t_0}(K,T,S_{t_0}) \leq \gamma^b_{t_0}(K,T,S_{t_0},p) \leq \frac{p}{1-p} C_{t_0}(K,T,S_{t_0}), \tag{25}$$

and

$$p(S_{t_0} - B_{t_0,T}K)_+ \leq \gamma^b_{t_0}(K,T,S_{t_0},p).$$

Proof. Exactly as the proof of Proposition 3.3. □

Remark. The proofs for existence and uniqueness of prices are easily modified to cover general European-style cash-settled OOO options. A solution of the valuation equation (19) exists and is unique for p small enough, if

(i) $x \mapsto x - \kappa + ph(x)$ (if the options are bought, or $x \mapsto x + \kappa - ph(x)$, if the options are sold) is invertible. This holds, if h satisfies a Lipschitz condition $|h(x_2) - h(x_1)| \leq L(h)|x_2 - x_1|$ and $pL(h) < 1$. Denote the inverse by f_κ.

(ii) $\mathbb{E}^Q[h \circ f_0(S_T) \mid \mathcal{F}_{t_0}] > 0$ a.e..

(iii) f_κ is Lipschitz in κ, i.e. $|f_{\kappa_2}(s) - f_{\kappa_1}(s)| \leq \tilde{L}(f)|\kappa_2 - \kappa_1|$ and $pL(h)\tilde{L}(f)B_{t_0,T} < 1 \quad \forall s$.

4. Subsequent Effects on the Stock Price Process

In a footnote, Galai and Schneller (1978, p. 1336) point out that if the value of a company's assets at maturity is lognormally distributed, the distribution of the value of the company after issuing warrants will not be lognormal. When the distribution of the company's market value at maturity of the OOO option changes, arbitrage considerations imply that the value before this date (and, thus, the stochastic process of the stock price) also changes. Schulz and Trautmann (1994) analyze effects of warrants issuance on the stock price process and find that volatility decreases as a result of warrants issuance. In Section 2, we have shown that warrants can be seen as a special case of our general problem.

We summarize the effects of transactions in OOO options on the dynamics of the underlying within the framework of one-dimensional diffusion processes. For simplicity of exposition, we assume throughout this section that the risk-free interest rate is constant.

Let $(\Omega, \mathcal{F}, Q)$ denote a probability space, $(W_t)_{0\leq t\leq T}$ a standard Brownian motion under Q, $(\mathcal{F}_t)_{0\leq t\leq T}$ its natural filtration and $(S_t)_{0\leq t\leq T}$ a diffusion adapted to $(\mathcal{F}_t)_{0\leq t\leq T}$ such that the discounted process $(\tilde{S}_t)_{0\leq t\leq T}$ is a positive square integrable martingale under Q. Furthermore, we assume that

$$\tilde{S}_t = \tilde{S}_0 + \int_0^t \tilde{S}_u \sigma(S_u, u) dW_u,$$

where $(\tilde{S}_t\sigma(S_t,t))_{0\leq t\leq T}$ is adapted and $\mathbb{E}^Q[\int_0^T (\tilde{S}_t\sigma(S_t,t))^2 \mathrm{d}t] < \infty$.

Proposition 4.1. *Let $\mathbb{E}^Q[g^2(S_T)] < \infty$, where $g = h \circ f_\kappa$ and let $(X_t)_{0\leq t\leq T}$ be defined by (10) for $t_0 \leq t \leq T$ and by $X_t = S_t$ for $0 \leq t < t_0$. An adapted process $(H_t)_{t_0\leq t\leq T}$ exists, such that $\mathbb{E}^Q[\sup_{0\leq t\leq T} H_t^2] < \infty$ and*

$$X_t = \begin{cases} S_{t_0} + \mathrm{e}^{rt} \int_{t_0}^t \left(\tilde{S}_u\sigma(S_u, u) + H_u\right) dW_u & t_0 \leq t \leq T, \\ S_t & 0 \leq t < t_0. \end{cases}$$

The discounted process $(\tilde{X}_t)_{0\leq t\leq T}$ is a continuous square-integrable martingale under Q.

Proof. Let $\tilde{M}_t = \mathbb{E}^Q[pg(S_T)\mathrm{e}^{-rT} \mid \mathcal{F}_t]$. $(\tilde{M}_t)_{t_0\leq t\leq T}$ is a square-integrable martingale, which admits a representation

$$\tilde{M}_t = \tilde{M}_{t_0} + \int_{t_0}^t H_u dW_u,$$

where (H_u) is an adapted process satisfying $\mathbb{E}^Q[\sup_{0\leq t\leq T} H_t^2] < \infty$, so that for $t \geq t_0$

$$\tilde{X}_t = \tilde{S}_t - \kappa\mathrm{e}^{-rt_0} + \tilde{M}_{t_0} + \int_{t_0}^t H_u dW_u.$$

$\tilde{M}_{t_0} = \kappa\mathrm{e}^{-rt_0}$ concludes the proof. □

Example 4.1. Let the company buy p cash-settled OOO put options on its market value, which is modeled as a geometric Brownian motion (Black-Scholes model). Then $d\tilde{S}_t = \tilde{S}_t\sigma dW_t$, where $(W_t)_{0\leq t\leq T}$ is a Brownian motion under Q. Furthermore, $H_t = -\sigma\tilde{S}_t(\Phi(-d_1(S_t))p/(1+p) + 1/(1+p))$, with Φ the c.d.f. of the standard normal distribution and $d_1(x) = (\log(x/(K + \mathrm{e}^{r(T-t_0)}\pi)) + (r + \sigma^2/2)(T-t))/\sqrt{\sigma^2(T-t)}$. Thus for $t_0 \leq t \leq T$,

$$X_t = S_{t_0} + \int_{t_0}^t \mathrm{e}^{r(t-u)} S_u\sigma\left(1 - \Phi(-d_1(S_u))p/(1+p)\right) dW_u.$$

□

Example 4.1 shows that the purchase of OOO puts leads to a decrease in stock price volatility (here, $H_t < 0$). The same effect occurs if OOO calls are sold by the company. This corresponds to the findings of Schulz and Trautmann (1994) for the special case of warrants. The remaining two positions in OOO options (calls bought and puts sold) lead to an increase in stock price volatility.

Under the diffusion models considered above, prices of derivatives with convex payoff functions are monotone functions of the volatility of the underlying. In the following proposition we prove an analogous result for derivatives defined on underlyings S and X. Denote by $(H_t)_{t_0 \le t \le T}$ the process defined in Proposition 4.1. From Proposition 4.1, the diffusion coefficient of X_t is driven by the diffusion coefficient of $\tilde{S}_t$ and H_t. Let $\hat{\sigma}(S_t, t)$ be given by

$$H_t = \tilde{S}_t \hat{\sigma}(S_t, t). \tag{26}$$

The idea of the following result is that if $\hat{\sigma} > 0$, X has greater volatility than S, and prices of options (with convex payoff) on X will be higher than on S. If, however, $-\sigma(S_t, t) < \hat{\sigma}(S_t, t) < 0$, then the prices of options on S will be higher. For the special case of options on stocks of warrant-issuing firms, this effect has been explored by Hanke and Pötzelberger (2002).

Consider the Black-Scholes model for (S_t) in Example 4.1, where the company buys p cash-settled OOO put options on its market value. Since

$$\hat{\sigma}(S_t, t) = -\sigma(\Phi(-d_1(S_t))p/(1+p) + 1/(1+p)),$$

we have $-\sigma < \hat{\sigma} < 0$ and thus Proposition 4.2 tells us that options on S are more expensive than options on X.

Proposition 4.2. *Let a derivative be defined by a payment function φ which depends on the value of the underlying at time T^* only. Denote by $V_{t,S}$ and by $V_{t,X}$ the prices of derivatives written on the underlying S, given $S_t = z$ and on X, given $X_t = z$, respectively. Let $(H_t)_{t_0 \le t \le T}$ satisfy (26) and let the assumptions of Proposition 4.1 hold. Let the ordinary differential equations*

$$\frac{\mathrm{d}\tau_t}{\mathrm{d}t} = \frac{1}{W_t^2 \sigma(W_t, \tau_t)^2}$$

and

$$\frac{\mathrm{d}\tau_t}{\mathrm{d}t} = \frac{1}{W_t^2 (\sigma(W_t, \tau_t) + \hat{\sigma}(W_t, \tau_t))^2}$$

have, almost surely, unique strictly increasing solutions which explode when W_t first reaches zero.

1. *Let $-\sigma(x,t) < \hat{\sigma}(x,t) \le 0$ for all x and all $t_0 \le t \le T$. If φ is convex, then $V_{t,X} \le V_{t,S}$ for all t.*

2. *Let $\hat{\sigma}(x,t) \ge 0$ for all x and all $t_0 \le t \le T$. If φ is convex, then $V_{t,X} \ge V_{t,S}$ for all t.*

Proof. It is sufficient to prove the proposition for $T^* > t_0$. Fix $t \leq T^*$. Then $X_v = S_{t_0} + \mathrm{e}^{rv} \int_{t_0}^{v} (\tilde{S}_u \sigma(S_u, u) + H_u) dW_u$ for $t \leq v \leq T^*$. Note that $(\tilde{X}_t)$ is a square-integrable martingale for which Hobson (1998), Theorem 2.1 is applicable. Therefore, if $-\sigma(x,u) < \hat{\sigma}(x,u) \leq 0$ and if φ is convex,

$$\tilde{V}_{t,X} = \mathbb{E}^Q[\varphi(X_{T^*})\mathrm{e}^{-rT^*} \mid \mathcal{F}_t] \leq \mathbb{E}^Q[\varphi(S_{T^*})\mathrm{e}^{-rT^*} \mid \mathcal{F}_t] = \tilde{V}_{t,S}.$$

Analogously, if $\hat{\sigma}(x,t) \geq 0$ for all x and all $t_0 \leq t \leq T$ and if φ is convex,

$$\tilde{V}_{t,X} = \mathbb{E}^Q[\varphi(X_{T^*})\mathrm{e}^{-rT^*} \mid \mathcal{F}_t] \geq \mathbb{E}^Q[\varphi(S_{T^*})\mathrm{e}^{-rT^*} \mid \mathcal{F}_t] = \tilde{V}_{t,S}.$$

□

Remark. Stock prices are often modeled using Markov processes. When a company takes positions in OOO options, the Markov property is lost. The stock price at a certain point in time *after* the option transaction depends on the value of the company's productive assets, the terms of the option and, in particular, the time of the option transaction.

5. Numerical Examples

In this section we present two examples. The first example shows the differences in option prices between standard options (bought or sold by outside investors) and OOO options (bought or sold by the company itself). The second example demonstrates the effect of one OOO option transaction on the pricing of another OOO option transaction that occurs later in time. This is an indirect effect of the change in the stock price process.

Example 5.1. In this example, we assume a standard Black-Scholes world, i.e., the process (S_t) is a geometric Brownian motion under Q with $S_0 = 100$, $\sigma = 0.4$, and $r = 0.03$. Furthermore, the strike price $K = 100$ and T, the expiration time, is 1 year. From the four basic OOO option positions analyzed in Section 2, we select two for reasons of economic significance: Short OOO call positions correspond to employee stock options, and long OOO put positions might be used as insurance against bankruptcy (an idea put forward by Murphy (1998) in a slightly different context).

The solid lines in Figures 3 and 4 show the values of options according to the Black-Scholes formula. Using the Black-Scholes formula implies neglecting the fact that the company itself buys or sells options on its own value. The dotted lines show the correct values of p OOO options (taking into account that these options are sold (bought) by the same company on whose stocks the options are written).

Example 5.2. Let $(S_t)_{t\in[0,1]}$ denote the value of the company without any transactions in OOO options. S_t is modeled as a geometric Brownian motion under Q with $S_0 = 100$, $\sigma = 0.4$ and $r = 0.03$. At $t = 0$ $(=: t_0)$, the company buys p_0 cash-settled European OOO put options on its market value X, with strike price $K = 100$ and expiration time $T = 1$. At $t = 1/2$ $(=: t_1)$, it buys another p_1 cash-settled OOO put options on its value,

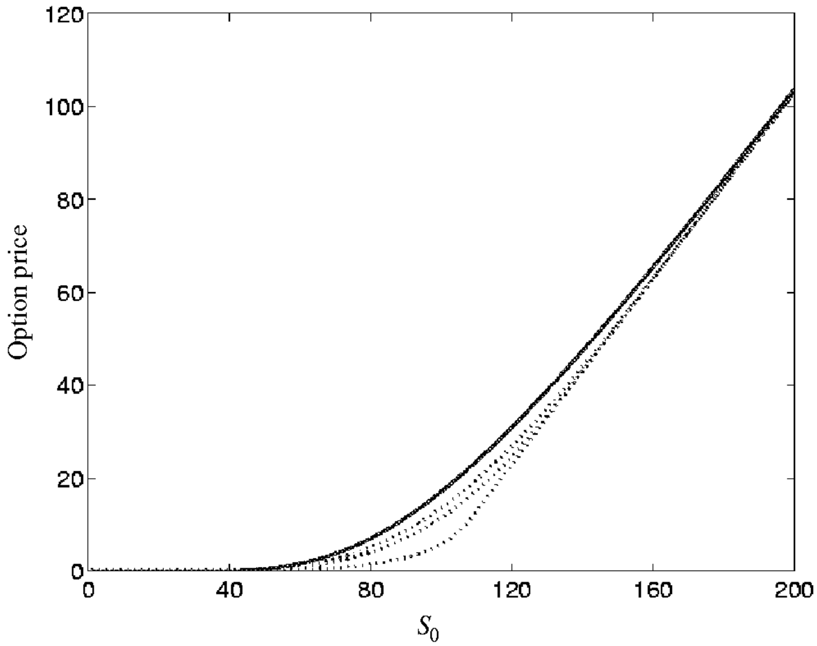

Figure 3. Deviation of the price of p OOO call options sold by the company (dotted lines) from the Black-Scholes price (solid line), $p = 0.5, 1, 5$.

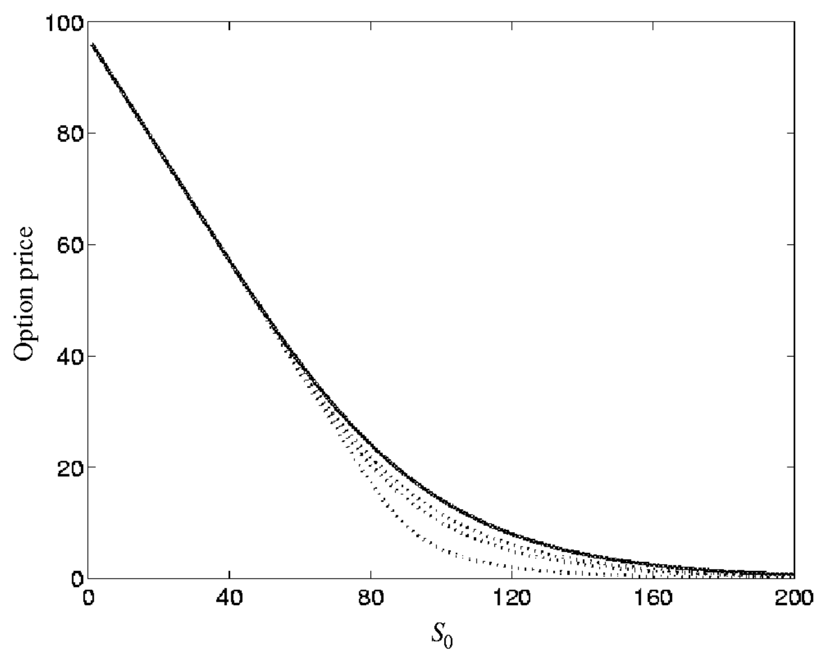

Figure 4. Deviation of the price of p OOO put options bought by the company (dotted lines) from the Black-Scholes price (solid line), $p = 0.5, 1, 5$.

again with strike price $K = 100$ and expiration date $T = 1$. We denote by $\pi_1(S_{t_1})$ the price of these p_1 put options at time t_1.

After the purchase of p_0 puts at time t_0, the process for the market value of the company becomes

$$X_t = S_t - B_{t,t_0}\pi_0 + \frac{p_0}{1+p_0}P_t(K + B_{T,t_0}\pi_0, T, S_t) \tag{27}$$

with

$$\pi_0 = \frac{p_0}{1+p_0}P_{t_0}(K + B_{T,t_0}\pi_0, T, S_{t_0}).$$

For the valuation of the p_1 puts bought at time t_1, note that X_t has already changed as shown in (27). Therefore, X_t now plays the role of S_t in the valuation equation, and

$$\pi_1 = \frac{p_1}{1+p_1}\mathbb{E}^Q\left[B_{t_1,T}(K + B_{T,t_1}\pi_1 - X_T)_+|\mathcal{F}_{t_1}\right].$$

Substituting for X_T and taking cases ultimately leads to

$$\pi_1 = \frac{p_1}{1+p_1}P_{t_1}(K + B_{T,t_1}\pi_1 + B_{T,t_0}\pi_0, T, S_{t_1}) - \frac{p_1}{1+p_1}\frac{p_0}{1+p_0}P_{t_1}(K + B_{T,t_0}\pi_0, T, S_{t_1}).$$

Figure 5 exhibits three prices of put options. The solid line is the price of $p_1 = 1$ standard put options on S (i.e., ignoring the feedback effect). The dotted/dashed lines are the prices of $p_1 = 1$ cash-settled OOO put options on X. The dotted line (that is close to, but just below the solid line) represents the price when the purchase of p_0 put options at time t_0 is neglected. The dashed line is the graph of $\pi_1(S_{t_1})$ for $p_0 = p_1 = 1$ if the previous transaction at t_0 is properly taken into account.

6. Practical Implications

The most important practical implication arises from Assumption 1 in Section 2: Market participants are fully informed about a company's positions in OOO options. In particular, the respective counterparty of a corporate OOO option transaction knows that it is dealing with the company itself, as opposed to a third party. This assumption is clearly violated if – as is the case in many legislations – companies are not legally prohibited from taking positions in exchange-traded options, even if these options are written on their own stocks. As soon as the company is one of the parties to such a transaction, a standard exchange-traded option becomes a OOO option. Rational counterparties would, as we have seen before, pay or demand a different price if they knew who their counterparty really is. Especially for long call and short put positions in OOO options, the company has an incentive to arrange such deals via "anonymous" exchanges, rather than direct negotiations with a financial institution (and, thus, conceal its identity from its counterparty): In these cases, the company can reap arbitrage profits, because buyers of puts would pay more than, and sellers of calls would charge less than the fair (no-arbitrage) value. This can only be prevented by prohibiting corporate transactions in exchange-traded options written on their own stocks.

Another important caveat arises from Assumption 3 in Section 2: There are no signaling effects associated with transactions in OOO options. However, one can hardly imagine a financial institution, approached by a company willing to buy OOO puts, not thinking twice about possible hidden motives of that firm. In practice, convincing the

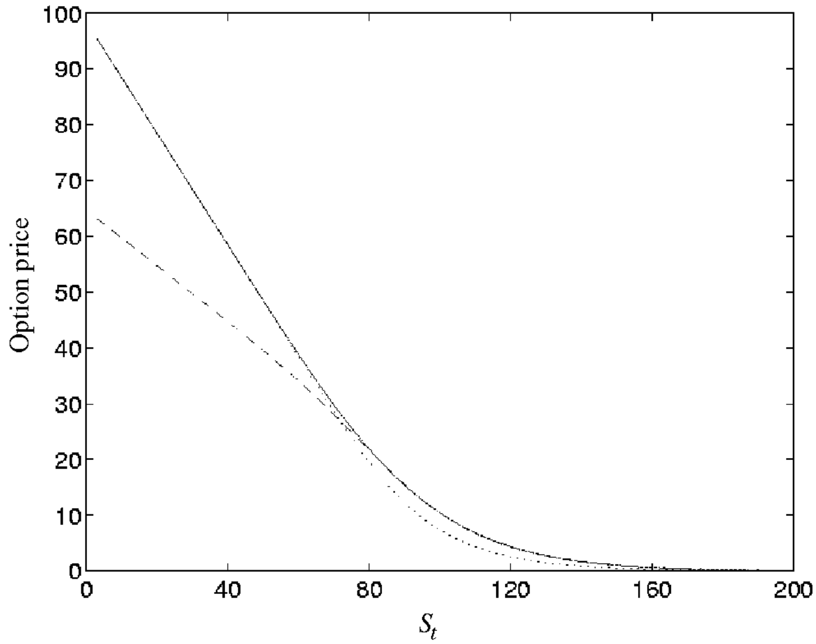

Figure 5. Effect of the change in the stock price process: The company buys one OOO put on X_T at $t = 0$ and another OOO put at $t = 0.5$. The dashed line is the correct price for the second OOO put. The dotted line is the OOO option price (ignoring the existence of the first OOO put). The solid line is the Black–Scholes price (ignoring both the feedback effect and the existence of the first OOO put). Note the large deviation resulting from neglecting the effect of the first put on pricing the second.

financial institution that the motivation for this is purely "technical" (e.g., a cheaper means of insuring against bankruptcy, as suggested by Murphy (1998)), as opposed to a consequence of information asymmetries, might be very difficult.

7. Summary and Directions for Further Research

In this paper, we have analyzed effects occurring when a company takes positions in cash-settled options written on this company's stocks. We have shown how standard pricing models have to be modified in this case and have provided valuation equations for generalized European-style payoff functions. Then, we have shown how the stochastic process of the stock price changes as a consequence of transactions in cash-settled OOO options due to the resulting feedback between stock price and option price. Using numerical examples, we have demonstrated the extent of this effect, and discussed several practical aspects.

In further research work, we will analyze OOO options with physical delivery on exercise as well as American-style OOO options. Another related paper (Hanke/Pötzelberger (2002)) deals with the changes in the stock price process occurring when a company issues warrants and analyzes subsequent changes in the prices of traded options written on companies which issue warrants.

References

Carpenter J N 1998 The exercise and valuation of executive stock options *J. Financial Economics* **48** 127–158

Frey R and A Stremme 1997 Market volatility and feedback effects from dynamic hedging *Mathematical Finance* **7** 351–374

Galai D and Schneller M I 1978 Pricing of warrants and the value of the firm *J. Finance* **33** 1333–1342

Hanke M and Pötzelberger K 2002 Consistent pricing of warrants and traded options *Rev. Financial Economics* **11** 63–77

Heath D, E Platen and M Schweizer 2001 A comparison of two quadratic approaches to hedging in incomplete markets *Mathematical Finance* **11** 385–413

Hobson D G 1998 Volatility misspecification, option pricing and superreplication via coupling *Ann. Applied Probability* **8** 193–205

Murphy A 1998 Buying put options to insure against bankruptcy *J. Financial Engineering* **6** 307–319

Platen E and M Schweizer 1998 On feedback effects from hedging derivatives *Mathematical Finance* **8** 67–84

Schulz G U and Trautmann S 1994 Robustness of option-like warrant valuation *J. Banking and Finance* **18** 841–859