

Price Dynamics for Continuously Produced Storable Commodities: Competitive and Monopolistic Markets*

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March 16, 2003

*We would like to thank Elizabeth Whalley for her extensive comments. We also thank Dwight Barkley for his helpful comments on the numerical work. All errors are solely our responsibility.

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Abstract

We develop a new structural model in continuous time for storable commodities, which is primarily developed for storable energy commodities, such as natural gas and oil. Our model formulation establishes a link between the discrete time models developed in the past for agricultural commodities and the reduced form models recently developed for energy storable commodities. Particularly, we formulate and solve the model following the structural models framework but we simultaneously incorporate characteristics of reduced form models, namely the mean-reversion of spot prices. Such mean-reversion is incorporated in our model through the definition of the exogenous supply as an Ornstein-Uhlenbeck process. The explicit definition of the microeconomic assumptions of our model makes it flexible and relatively easy to modify. We build our model having storable energy commodities in mind but it would be also well suited for other storable commodities after a few small changes. We formulate our model in a general framework and later specify it for two distinct economies - pure competition and storage monopoly. We use stochastic dynamic programming approach and since an analytical solution is not possible to obtain, we get a numerical solution to the problem. We analyze and compare the results for both economic scenarios. Our main results are in accordance with the theory of storage: the presence of storage in the commodities market smoothes the prices behavior compared to the case where no storage is possible. This result is stronger in the case of pure competition than in the case of monopolistic storage.

Keywords: storage, structural model, price dynamics, continuous-time, equilibrium.

JEL classification: G11 and G13.

1 Introduction

We present a continuous-time structural model for storable commodity prices, where supply has mean-reversion characteristics. The model is initially developed under a general framework and later it departs into two distinct forms where each of them represents a different economic market scenario - perfect competition and monopolistic storage. It has been primarily developed for natural gas and oil but it is also well suited for many other non-perishable commodities.

The existing literature on modeling commodity prices has been disjoint and therefore, the main attraction of our approach is the fact that we establish a link between the two major categories of the literature - discrete-time structural commodity price models and continuous-time reduced form models. The latter have been used in the particular context of energy commodities, such as oil. Specifically, we use some of the characteristics of the reduced-form models but we draw on a structural model formulation in the fashion of those originally developed to study agricultural commodity prices.

The structural models are explicitly derived from economic primitives. The essence of this approach is the solution of functional equations to derive numerical approximations for the functions relating supply and availability. The core articles in this class include Scheinkman and Schectman (1983) , Williams and Wright (1991), Deaton and Laroque (1996), Chamber and Bailey (1996), Pirrong (1998) and Routledge, Seppi and Spatt (2000). These papers consider commodity equilibrium price models which are based on the theory of storage of Kaldor (1939), Working (1949) and Telser (1958) , and explain the convenience yield in terms of an embedded timing option (Routledge, Seppi and Spatt (2000)). These studies develop discrete-time, infinite horizon spot and forward price equilibrium models in which a single homogeneous commodity is traded in a competitive market at successive discrete dates. Within this economic context, storage is purely speculative, that is, the single motive of the risk-neutral traders for holding inventory is trading profit.

The recent development of traded options on energy has stimulated the development of reduced form models of energy commodities spot prices. In essence, these models are adapted from other standard financial assets. What makes these models particularly attractive is the fact that they are easily calibrated to the real data and demand a small computational effort to be implemented. The reduced form models treat the spot price and the convenience yield as separate factors. The spot price follows a lognormal, mean-reverting diffusion process and the convenience yield is an exogenous stochastic process correlated to the spot price, like an exogenous dividend process. This perspective is standard in both the literature and the practice of energy commodities options pricing and provide powerful tools for derivative pricing and hedging. Leading works within this framework include Gibson and Schwartz (1990), Amin, Ng and Pirrong (1995) , Schwartz (1997), and Hilliard and Reis (1998). However this approach has inadequacies.

First, these models generate inconsistencies between the spot price and the convenience yield. The other drawback is that the spot prices and convenience yield are modeled as having constant correlation, which is unlikely to happen in reality, due to the dependency of the correlation on inventory (Routledge, Seppi and Spatt (2000)).

In this paper, we build our model on the microeconomics of supply, demand and storage like the structural models. On the other hand, as in the reduced form models, we consider a continuous-time framework and make use of a mean-reverting continuous-time stochastic process. However, in our model, the mean-reversion occurs in the exogenous supply and not in the spot price process itself. Note that we can interpret this mean-reverting process as the difference between the exogenous supply and the stochastic demand in the market. Within the pure competitive context, we suppose that a single homogeneous commodity is continuously traded in a competitive, risk-neutral market, where storage is purely speculative over a finite-time horizon. We then extend the above formulation to a market structure where storage is monopolistic but production and sales remain purely competitive. Given the complexity of the problem, we apply a stochastic dynamic programming formulation which yields a dynamic optimal storage policy in continuous-time. The resulting inventory policy determines the market price within each of the market structures considered.

Our approach has several advantages. It introduces a continuous-time structural model that draws on specific microeconomic assumptions of the market environment. Accordingly, it builds on the structural models, but employs a continuous-time framework. Like the structural models, it explicitly establishes the link between storage management and price dynamics. Another attraction of our models is the employment of a flexible framework which allows for different extensions of the model by defining alternative microeconomic market conditions, and therefore enables the model to be adapted to other commodities. This flexibility enabled us to consider both pure competition and monopoly over storage formulations. The study of these two market environments is particularly relevant for the natural gas market since it has been evolving from a regulated to an unregulated market environment.

Our main results are that the existence of storage smoothes the market prices, reducing the dispersion of the price dynamics. However, this effect is much stronger in the case of the competitive storage economy than in the monopolistic one. Also, the magnitude of storage availability affects the extent to which the storage agents are able to implement the optimal storage management policy.

Section 2 formulates the model and describes the solution method. Section 3 presents the results. In section 4 we present the conclusions. Appendix A gives the derivation of the stochastic dynamic programming equations used to obtain the solution to the model.

2 Solving for Storage Equilibrium

Our analysis builds on and extends the discrete-time framework suggested by Williams and Wright (1991), Chapter 3, where a discrete-time general model for commodities in a pure competitive market is formulated using discrete-time dynamic programming. The state variable is the amount on hand in period t , furnished from production and/or previous storage and the decision variable is the inventory level. The storage decisions are made by a single identity, the social planner. The planner's problem in the current period, t , is to select the current storage that will maximize the discounted stream of expected future surplus. We adopt the same formulation but we adapt it to a continuous-time framework by assuming that the state variables are driven by continuous-time processes. In particular, the state variables in our models are the endogenous rate of change in storage level and the exogenous stochastic rate of supply, which is driven by a mean-reversing stochastic process. Now, the decision variable is the rate of change in storage level, which can be either positive or negative. Within the monopolistic storage context, we further modify the original formulation by considering that the decisions are made by the monopolistic storage agent which maximizes the discounted stream of expected future cash-flows. For each of the competitive and monopolistic storage market scenarios, we determine a storage policy by specifying the rate of storage at each moment in the future for each possible state of the world. The optimality of storing involves the determination of the storage rate that best serves the purposes of the economic agents that take the decisions within each of the two market environments. At each moment in time, the total aggregate rate of consumption of the commodity is calculated as the difference between the exogenous rate of supply and the endogenous rate of change in inventory level. Having the aggregate consumption, we finally calculate the price using the inverse demand function.

2.1 Model Formulation and Solution

Both models are developed using the same basic framework. In one case we assume that the market (including storage) is perfectly competitive; in the other we assume that storage (only) is monopolistic. In the competitive equilibrium, we assume that the number of firms in the storage industry is sufficiently large for each to be a price taker. The storage decisions are made by a single identity, the "invisible hand". Under monopolistic storage, consumers can deal directly with producers through the market but neither group can store on its own. Only one firm has the right or the technology to store the commodity. A monopolistic firm does not extract its extra profits by holding the commodity off the market to keep the price high, because it is not the only source for consumers, as the producers also continuously supplying the market. Likewise, the firm competes with consumers for any quantity it purchases on the market.

We introduce the model under a general formulation and later on diverge into the two distinct market scenarios.

The general assumptions of the model are as follows:

- A single homogeneous commodity is produced and traded in continuous-time, over a finite-time horizon T ;
- Storage is purely speculative, whereby inventory decisions are driven by the single motive of trading profit;
- The supply has zero elasticity;
- All the agents of the model are risk-neutral;
- The marginal storage cost, k , is constant; the storage cost is $k \times s$ per unit of time, where s is the current storage level;
- The one-period risk-free interest rate, $r \geq 0$ is constant.

We consider two state variables: the exogenous supply rate and the inventory level. The exogenous supply rate, z_t , is given by:

$$dz_t = \alpha(\bar{z} - z_t)dt + \sigma dB_t, \quad t \geq 0 \quad (1)$$

where:

- α is the speed of mean-reversion;
- \bar{z} is the long-run mean, that is, the level to which z reverts as t goes to infinity;
- σ is the (constant) volatility;
- B_t is a standard Wiener process.

The aggregate storage level, s , is a fully controllable endogenous state variable and satisfies:

$$ds = u(s, z, t)dt, \quad t \geq 0, \quad s \geq 0 \quad (2)$$

where u represents the rate of storage and is the decision variable in our problem. At each time t , the rate at which the commodity is stored depends on the amount already in storage, s , and on the exogenous supply, z .

Note that the decision $u(\cdot)$ is a function on $[0, T]$, which we call the *inventory management plan*. If the inventory capacity is $b > 0$, then the inventory level $s(t)$ must satisfy the constraint:

$$0 \leq s(t) \leq b \quad (3)$$

since negative storage is not allowed. On the other hand, if $z(t)$ is the supply rate at time t , then $u(\cdot)$ must not exceed this rate, that is:

$$u(t) \leq z(t) \quad (4)$$

Constraints (3) and (4) imply that the optimal storage rate, u^* , belongs to

$[u_{\min}, u_{\max}]$ where the values u_{\min} and u_{\max} are such that these two constraints are satisfied. Any inventory management plan that satisfies these conditions is called an *admissible plan*. The total rate of consumption in the market, q , establishes the relationship between the state variables defined above and satisfies the equilibrium condition:

$$q = z - u \quad (5)$$

Additionally, the market price (or inverse demand function) is given by $p(q)$, where $\frac{\partial p}{\partial q} < 0$.

We consider a finite-time horizon T , at which there is no carryover and we work backwards in time. The following functional is then to be maximized:

$$V(s_t, z_t, t; u(\cdot)) = \left\{ E_t \int_t^T e^{-r(l-t)} L(s_l, z_l, u_l, l) dl + \Psi(s_T, z_T) \mid s = S, z = Z \right\} \quad (6)$$

over all the admissible plans where $L(s_t, z_t, u_t, t)$ is the instantaneous profit rate and $\Psi(s_T, z_T)$ is the *salvage value* of having s_T and z_T as states at final time T . Without loss of generality, we consider $\Psi(s_T, z_T) = 0$, that is, the *Lagrange* form of the optimal control problem. The crucial difference between the pure competition and the monopolistic storage problem formulation consists in the definition of L , which we will discuss later.

To obtain the solution to the problem we use the dynamic programming approach. Accordingly, we need to maximize the value function, V , in order to obtain the optimal set of carryover decisions through time. We apply the principle of optimality (Bellman, 1957) to (6) and obtain a dynamic programming equation of the form (see derivation in Appendix A):

$$-\frac{\partial}{\partial t}V(s, z, t) - H(s, z, V_s, V_z, V_{zz}) = 0 \quad (7)$$

where:

$$H(s, z, V_s, V_z, V_{zz}) = \sup_{u \in [u_{\min}, u_{\max}]} \left\{ \begin{array}{l} L(s, z, u, t) + uV_s(s, z, t) + \\ \alpha(\bar{z} - z)V_z(s, z, t) + \frac{1}{2}\sigma^2V_{zz}(s, z, t) - rV(s, z, t) \end{array} \right\} \quad (8)$$

for the value function $V(s, z, t)$ with the boundary condition $V(s, z, T) = 0$. This yields the optimal u^* . Note that u^* needs to be such that the storage constraints are not violated, that is $u_{Min} \leq u^* \leq u_{Max}$. If u^r represents the maximum of the above dynamic programming equation, then:

$$u^* = u_{\max}, \text{ if } u_{\max} \leq u^r; \quad (9)$$

$$u^* = u^r, \text{ if } u_{\min} \leq u^r \leq u_{\max}; \quad (10)$$

$$u^* = u_{\min}, \text{ if } u^r \leq u_{\min}; \quad (11)$$

Finally, the current price is given by:

$$p(q) = p(z - u^*) \quad (12)$$

In what follows, we separate the formulation into the competitive and the monopolistic scenarios. The distinction between these two formulations is imposed by the definition of the instantaneous profit rate, $L(s_t, z_t, u_t, t)$, which differs among these two contexts as mentioned above.

In the competitive market, the maximization is made from a social planner perspective. The social planner, in the current period t , aims to select the current rate of storage that will maximize the discounted stream of expected future surplus (see Gardner (1979) and Williams and Wright (1991) for an intuitive economic explanation). Let $p(q)$ represent the inverse demand function and also let:

$$f(x) = \int_0^x p(q) dq, \text{ for } x \geq 0, \quad (13)$$

Thus

$$L(s_t, z_t, u_t, t) = f(z_t - u_t) - ks_t \quad (14)$$

where k is the constant marginal and average physical cost per period;

Accordingly, the functional to be maximized is:

$$V(s_t, z_t, t; u) = \left\{ E_t \int_t^T (e^{-r(l-t)} f(z_l - u_l) - ks_l) dl + \Psi(s_T, z_T) \mid s = S, z = Z \right\} \quad (15)$$

and we obtain the following dynamic programming equation:

$$-\frac{\partial}{\partial t} V(s, z, t) - H(s, z, V_s, V_z, V_{zz}) = 0 \quad (16)$$

where

$$H(s, z, V_s, V_z, V_{zz}) = \sup_{u \in [u_{\min}, u_{\max}]} \left\{ \begin{aligned} & f(z - u) - ks + uV_s(s, z, t) + \\ & \alpha(\bar{z} - z)V_z(s, z, t) + \frac{1}{2}\sigma^2 V_{zz}(s, z, t) - rV(s, z, t) \end{aligned} \right\} \quad (17)$$

By differentiating the right hand side of the above equation with respect to u , we obtain the first order condition that allows us to find the maximum. Let $D(\cdot) = p^{-1}(x), x \geq 0$, represent the demand function. If we initially ignore the fact that u^* needs to satisfy the storage constraint, the (regular) maximum, u^r , is given by:

$$u^r = z - D(V_s^*) \quad (18)$$

Then, by taking into account the constraints given by (3) and (4) we obtain the optimal control value, u^* .

We now specify the problem for the case of a monopolistic storage economy. In this case, the monopolistic storage manager in the current period, t , aims to select the current rate of storage that will maximize the discounted stream of expected future cash flows generated by the management of his storage facility. The control variable, u , represents the rate of storage, that is, the absolute change in inventory level over an infinitesimally small interval of time; accordingly $-u$ is the amount he sells over each period to generate profits. The instantaneous rate of profit is given by:

$$L(s_t, z_t, u_t, t) = -u_t p(z_t - u_t) - ks_t \quad (19)$$

The functional to be maximized is:

$$V(s_t, z_t, t; u) = \left\{ E_t \int_t^T e^{-r(l-t)} (-u_l p(z_l - u_l) - k s_l) dl + \Psi(s_T, z_T) \mid s = S, z = Z \right\} \quad (20)$$

where $\Psi(s_T, z_T)$ represents the salvage value as before.

The resulting dynamic programming equation is:

$$-\frac{\partial}{\partial t} V(s, z, t) - H(s, z, V_s, V_z, V_{zz}) = 0 \quad (21)$$

where:

$$H(s, z, V_s, V_z, V_{zz}) = \sup_{u \in [u_{\min}, u_{\max}]} \left\{ \begin{array}{l} -up(z - u) - ks + uV_s(s, z, t) + \\ \alpha(\bar{z} - z)V_z(s, z, t) + \frac{1}{2}\sigma^2 V_{zz}(s, z, t) - rV(s, z, t) \end{array} \right\} \quad (22)$$

We obtain the regular control value, u^r by solving the first order condition of the right-hand side of the above equation given by:

$$-p(z - u) + up'(z - u) + V_s = 0 \quad (23)$$

where p' denotes the first order derivative of p in order to u . Depending on the inverse demand function considered, we might not obtain an explicit expression for u^r , therefore equation (22) might need to be solved numerically. We then obtain the optimal control value u^* taking into account the admissibility conditions.

It is important to observe that, since the Bellman equation is a backward equation, we must impose a final condition. Supposing that no salvage value remains at the final time¹:

$$V^*(s, z, T) = 0 \quad (24)$$

Accordingly, at final time T there is no carryover, therefore consumption equals supply. Moving backwards in time we obtain the set of optimal storage policies and the market price until the present time t .

We implement each problem numerically by solving the partial differential equations (PDEs) resulting from (17)-(18) and (22)-(23) respectively, using explicit finite differences.

¹Note that this assumption is for simplicity and not a restriction to the model.

3 Results

We present results for the present time, t_0 , and time horizon $T = 5$ years². We consider a long time horizon in order to obtain results for the steady state equilibrium. We first represent the optimal storage policy, $u^*(s, z, t)$ and then the resulting market price dynamics, $p(z - u^*)$, where s , z and t are the storage level, the exogenous supply level and time, respectively, as before. For illustrative purposes, we present the analysis of the case of a linear price function: $p(q) = p(z - u) = a - b(z - u)$. Table 1 represents the parameter values used for the implementation of the model and table 2 specifies the range for the annual supply rate distribution in the long run for these values and the inventory capacity considered³. Note that, in the long run, z has a stationary distribution, which is normal with mean $\mu_z = \bar{z} = 0.37$ and standard deviation $\sigma_z = \frac{\sigma}{\sqrt{2\alpha}} = 0.12$.

a	b	α	σ	\bar{z}	k	r
0.1	0.05	3	0.3	0.37	0.002	0.05

Table 1: Parameter used to obtain the numerical solutions of both competitive and monopolistic problems.

zMin	zMax	sMax
0	0.73	0.060

Table 2: zMin and zMax represent the lower and upper values of the grid for the exogenous supply rate, z . sMax represents the storage capacity.

In the absence of storage, the price is a function of the exogenous supply only, that is, $p(q) = p(z) = a - bz = 0.1 - 0.05z$, $z \in [0, 0.73]$ ⁴. Accordingly, in the long run, the price also follows a normal distribution with mean $\mu_p = 0.082$ and $\sigma_p = 0.006$.

Next, we present the results for the optimal storage policy and the resulting price within each of the competitive and the monopolistic market contexts. In particular, we emphasize the comparison between the price behavior in the absence of storage and in the presence of storage within each of the storage economies. We also compare the results between the pure competition and the monopolistic storage.

²We have also considered the possibility of developing this model for the steady state equilibrium by considering an infinite time horizon. However, the solution would be impossible to obtain without the knowledge of the boundary conditions for s and z .

³Although the exact results are unique to these parameters, all of the numerous parameter combinations studied produce similar qualitative results so those presented are representative.

⁴ $zMin = \mu_z - 3\sigma_z$ and $zMax = \mu_z + 3\sigma_z$.

3.1 Competitive Case

Figure 1 shows the optimal storage rate, u^* , as a function of both the storage level and the supply rate. Figure 2 shows the super-position of the price in the absence of storage and the price in the presence of storage in this economy as a function of both state variables. Figure 3 shows sections of the numerical difference between the two plots in Figure 2 at different fixed levels of storage, for greater clarity. This difference is calculated as the price in the presence of storage minus the price in the absence of storage.

The results are in accordance the intuition: the storage rate generally increases with the level of supply, approximately linearly. We also observe that the storage rate decreases with the storage level. At high levels of supply and when most of the storage capacity is used, the storage rate decreases significantly. It seems that when storage is close to its capacity and supply is very high there is little incentive to build inventory. This may be explained by the fact that storers do not expect to be able to sell at a profitable price in the near future.

By observing Figures 2 and 3 we conclude that the existence of storage stabilizes the prices. By looking at Figure 2, we observe that, if inventory space is available, prices are smoothed by the existence of storage. In particular, if the price is above the long-run mean⁵ (because supply is low), the existence of storage lowers the prices in relation to the original price function. On the other hand, if the price is below the long-run mean (because supply is high), the existence of storage increases the prices when compared with the original price function. We clearly conclude that storage affects the price dynamics by keeping the prices more stable and closer to the long-run mean, dampening down the slope of the original price function⁶. However, if the supply is high and most of the storage capacity is being used, the storage agents decrease significantly the rate of storage and the price behaves almost as if storage was inexistent. In the limit, at full storage capacity and high supply, the storers cannot build any further storage and therefore the prices behave as in the non-storage case. This result can be well observed in Figure 3.

⁵This is the long run-mean of the price distribution in the absence of storage.

⁶The volatility of the prices is directly proportional to the slope of the prices as a function of supply. Therefore damping down the slope means reducing the price volatility.

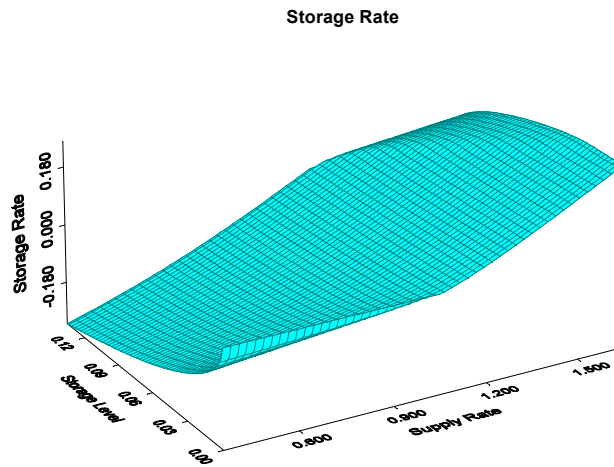


Figure 1: Competitive Case - Storage Rate, u , as a function of the two state variables storage, s , and exogenous supply rate, z .

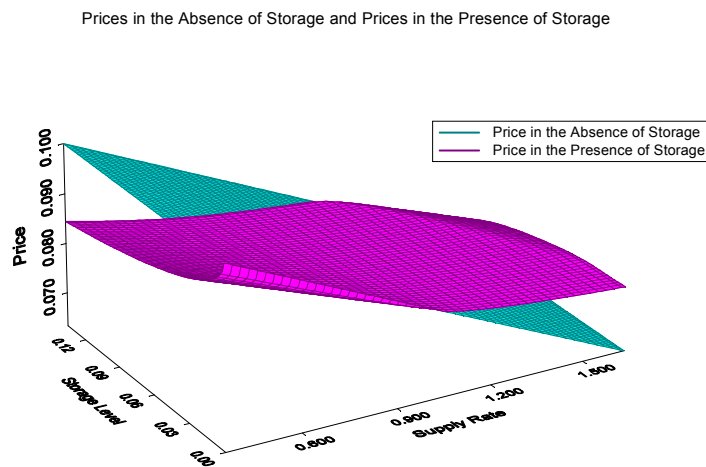


Figure 2: Competitive Case - Super-position of two graphs for the prices in the absence and in the presence of storage, respectively. The price is represented as a function of the two state variables storage, s , and exogenous supply rate, z .

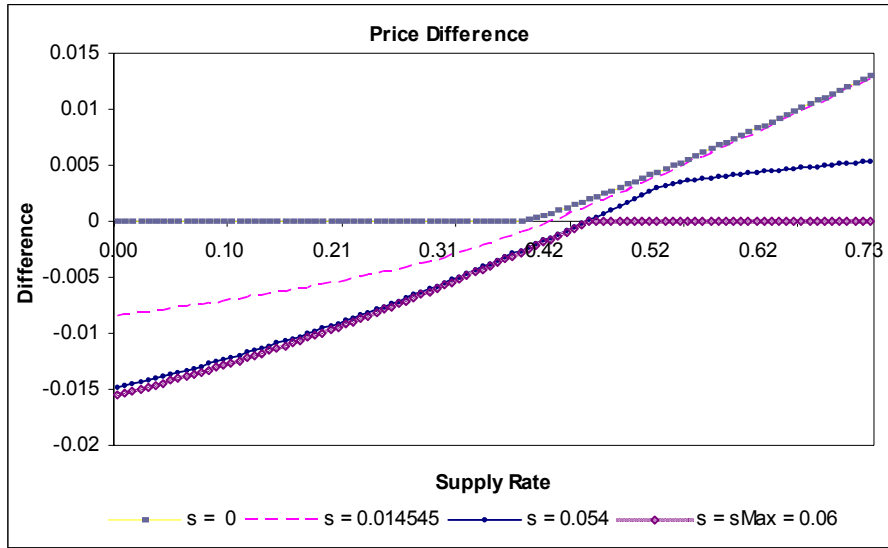


Figure 3: Competitive Case - Difference between prices in the presence of storage and prices in the absence of storage, at different fixed levels of storage.

3.2 Monopolistic Case

Figures 4, 5 and 6 represent the equivalent results for the monopolistic case. We observe that the storage rate increases with the rate of supply and decreases with the level of storage. When supply is high, the storage rate is relatively large and positive. However, for high values of the supply rate and when the storage is close to its capacity, the storage rate decreases. However, this latter result occurs in a significant lesser extent than in the competitive case.

By observing Figures 5 and 6 we see that the monopolistic storage agent's actions smooth prices as a function of the supply rate but to a lesser extent than in the competitive case. The monopolistic agent transacts less by comparison with the competitive agents and therefore his actions have a smaller effect on dampening the price movements.

The extent to which the competitive and the monopolistic actions affect the price behavior are significantly different. In the case of the pure competition, when supply is relatively high and significant storage capacity is available, prices are fairly stable. The exception occurs only when the supply is sufficiently high and the maximum storage capacity is almost reached, limiting further storage. This generates a greater supply in the market and therefore prices fall. In the monopolistic case, the storage policy does not have such a significant impact on the prices, dampening down the slope of the original price function to a significant smaller degree.

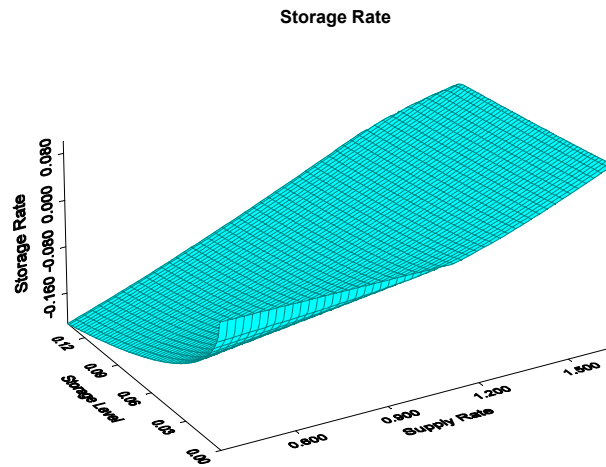


Figure 4: Monopolistic Case - Storage Rate, u as a function of the two state variables storage, s , and exogenous supply rate, z .

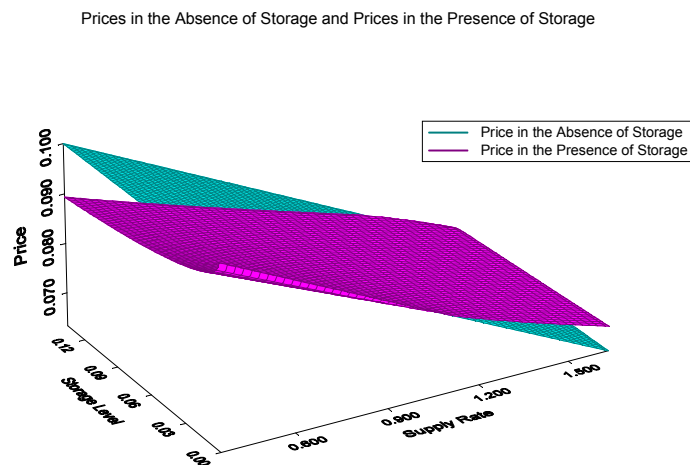


Figure 5: Monopolistic Case - Super-position of two price graphs, in the absence of storage and in the presence of storage, respectively. The price is represented as a function of the two state variables storage, s , and exogenous supply rate, z .

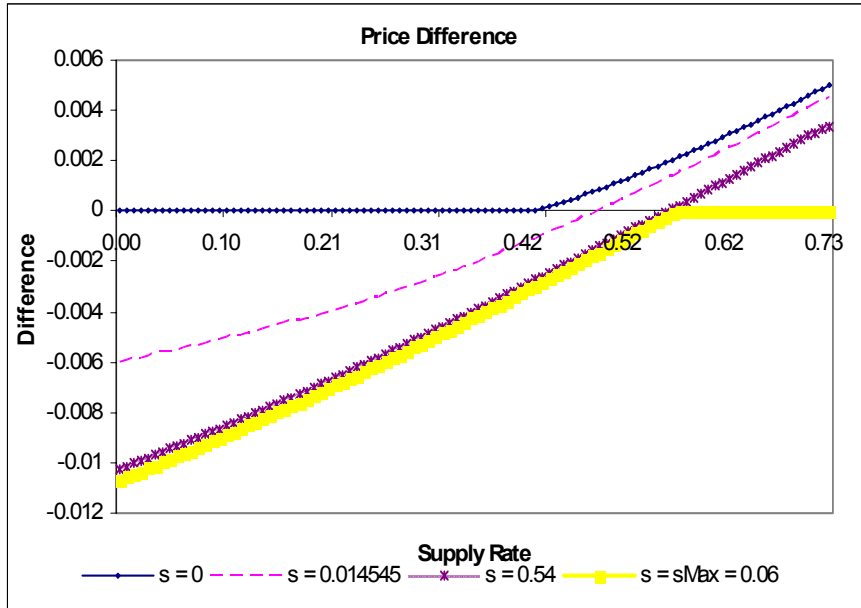


Figure 6: Competitive Case - Difference between prices in the presence of storage and prices in the absence of storage, at different fixed levels of storage.

4 Conclusions

This paper presents a new continuous-time structural price model suited for non-perishable storable commodities. Our model draws on the discrete-time structural models in the literature but also takes into account some features of the reduced form models recently developed in the literature for energy storable commodities. By using a mean-reverting process for the exogenous supply, it takes into account the mean-reverting characteristics of spot prices in energy markets.

The main results of the model are in accordance with the results predicted by the theory of storage. The presence of storage in the economy smoothes price behavior by reducing the tails of the price distribution from the no-storage case. This smoothing effect is more evident in the case of storage competition than in the case of monopolistic storage.

Another result is the effect of the storage capacity on the storage management plan and consequently on the price behavior. In particular, if the storage capacity is being fully used, the storage agents, in both economies, do not respond to price variations and consequently the price dynamics will follow the same process as in the absence of storage in the economy.

One of the main advantages of this model is its flexibility, by allowing different specifications of the microeconomic characteristics. By modifying price, supply or demand characteristics, we could easily extend this model to the study of other

commodities. Further work includes the development of reduced form models with the properties of this model.

A Derivation of the Stochastic Dynamic Programming Equation

We derive the stochastic dynamic programming resulting from maximizing the following functional:

$$J(s_t, z_t, t; u(\cdot)) = E_t \left\{ \int_t^T e^{-r(l-t)} L(s_l, z_l, u_l, l) dl + \Psi(s_T, z_T) \mid s = S, z = Z \right\} \quad (\text{A1})$$

over all the admissible plans where the state variables s and z satisfy the following transition equations:

- $dz_t = \alpha(\bar{z} - z_t)dt + \sigma dB_t$, $t \geq 0$; where B_t is a standard Wiener process defined on the underlying filtered probability space $(\Omega, F, \{F_t\}_{t \geq 0}, P)$.
- $ds = u(s, z, t)dt$, $t \geq 0$, $0 \leq s \leq b$;

and $L(s_t, z_t, u_t, t)$ is the instantaneous profit rate and $\Psi(s_T, z_T)$ is the salvage value of having s_T and z_T as states at final time T . Without loss of generality, we consider $\Psi(s_T, z_T) = 0$.

To solve the problem defined by equation (A1), let $V(s, z, t)$, known as the *Value Function* be the expected value of the objective function in (A1) form t to T when an optimal policy is followed from t to T , given $s_t = S$ and $z_t = Z$. Then, by the *principle of optimality*,

$$V(s, z, t) = \sup_{u \in [u_{\min}, u_{\max}]} E \left\{ L(s_t, z_t, u_t, t)dt + e^{-rdt} V(s + ds, z + dz, t + dt \mid s = S, z = Z) \right\} \quad (\text{A2})$$

where $[u_{\min}, u_{\max}]$ is defined in Section 2.

Multiplying both sides of the equation by e^{rdt} and noting that $e^{rdt} \simeq 1 + rh$ we obtain:

$$(1 + rdt) V(s, z, t) = \sup_{u \in [u_{\min}, u_{\max}]} E \left\{ L(s_t, z_t, u_t, t)dt + V(s + ds, z + dz, t + dt \mid s = S, z = Z) \right\} \quad (\text{A3})$$

That is:

$$\begin{aligned} rdt V(s, z, t) &= \sup_{u \in [u_{\min}, u_{\max}]} \left\{ L(s_t, z_t, u_t, t)dt + E_z (V(s + ds, z + dz, t + dt) - V(s, z, t) \mid s = S, z = Z) \right\} \\ &= \sup_{u \in [u_{\min}, u_{\max}]} \left\{ L(s_t, z_t, u_t, t)dt + E_z (dV(s, z, t) \mid s = S, z = Z) \right\} \end{aligned} \quad (\text{A4})$$

Applying Ito's calculus, we have:

$$dV(s, z, t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial z} dz + \frac{\partial^2 V}{\partial z^2} (dz)^2 \quad (\text{A5})$$

where ds and dz as above and $dz^2 = \sigma^2 dt$, which gives:

$$dV(s, z, t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial s} u dt + (\alpha(\bar{z} - z_t) dt + \sigma dB_t) \frac{\partial V}{\partial z} + \sigma^2 dt \frac{\partial^2 V}{\partial z^2}$$

which implies that

$$E(dV(s, z, t) | s = S, z = Z) = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial s} u + \alpha(\bar{z} - z) \frac{\partial V}{\partial z} + \sigma^2 \frac{\partial^2 V}{\partial z^2} \right) dt \quad (\text{A6})$$

Replacing (A6) into equation (A4) and dividing by dt gives:

$$rV(s, z, t) = \sup_{u \in [u_{\min}, u_{\max}]} \left\{ L(s_t, z_t, u_t, t) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial s} u + \alpha(\bar{z} - z) \frac{\partial V}{\partial z} + \sigma^2 \frac{\partial^2 V}{\partial z^2} \right\} \quad (\text{A7})$$

which is the stochastic dynamic programming equation we need to solve. Equation (A7) can be written in the following form:

$$-\frac{\partial}{\partial t} V(s, z, t) - H(s, z, V_s, V_z, V_{zz}) = 0 \quad (\text{A8})$$

where:

$$H(s, z, V_s, V_z, V_{zz}) = \sup_{u \in [u_{\min}, u_{\max}]} \left\{ \begin{aligned} &L(s, z, u, t) + u V_s(s, z, t) + \\ &\alpha(\bar{z} - z) V_z(s, z, t) + \frac{1}{2} \sigma^2 V_{zz}(s, z, t) - rV(s, z, t) \end{aligned} \right\} \quad (\text{A9})$$

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