Risk Aversion Elasticity and Risk Bearing

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Abstract

This paper applies the mean-standard deviation paradigm as to examine a widely used model of the hedging literature. As the standard hedging model satisfies a location and scale condition the mean-standard deviation technique provides more intuition for the revision of the firm's optimum risk bearing when price volatility changes. By introducing the elasticity of risk aversion to price risk we describe the interaction of price volatility and optimum hedge.

J.E.L. Classification: D11, D81, F31, F33, G22.

Key Words: Two-moment decision model, hedge ratio, elasticity of risk aversion.

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1. Introduction

Our study analyzes optimal hedging policy of a risk averse firm faced with an exogenous change in price risk. In contrast to the existing literature (see, e.g., Kimball, 1990, 1993) we focus on risk aversion elasticity to characterize the relationship between a change in risk and the optimum hedge ratio. Risk aversion elasticity is defined to be percentage change in risk aversion divided by percentage change in risk. The question how risk affects decision making is an important topic in many fields of economics, insurance and finance (Kimball, 1990; Briys et al., 1993; Broll et al. 1995; Wong 1997). The mean-standard deviation model provides a clear and straightforward economic intuition to the revision of the firm's optimum hedging when price risk changes.

We base our analysis on (μ, σ)-preferences. The (μ, σ)-criterion of decision making under uncertainty has experienced a growing attention in very recent contributions (Löffler, 1996; Bar-Shira and Finkelshtain, 1999; Eichner, 2000; Ormiston and Schlee, 2001; Battermann et al., 2002, Wagener, 2002). Important new insights are provided by two-moment decision models. For a (μ, σ)-risk averse firm our paper derives a clearcut relationship between changes in risk, the optimum hedge and risk aversion elasticity.

Meyer (1987) shows that if all prospects to be ranked are equal in distribution, except for location and scale (LS), then any expected utility ranking of all prospects can be based on the means and standard deviations of the alternatives' risky outcome. The standard hedging model actually implies that prospects are created through a shifting and scaling process.

Our study is organized as follows. First, the hedging model and the elasticity of risk aversion are introduced (section 2.1). Then the MS-approach is used to analyze in what manner risk, hedge ratio and risk aversion elasticity interact (section 2.2).

2. Risk aversion elasticity and risk effects

Consider a risk averse firm who owns a certain amount of an asset subject to price risk. At time 1 the firm sells x units at the prevailing spot market price p. This price is random at time 0. However, at time 0, the firm can hedge price risk by taking a short position, i.e. selling contracts H in the futures market. The hedge ratio is then defined as $h = H/x$. Note, that, by our definition the short position is positive.

We assume that the risk premium in the futures market is positive (= backwardation) and that the firm is (μ, σ)-risk averse. Furthermore, the current (= time 0) futures market price p_f is related to the delivery of one unit of the asset at time 1. Therefore our setting eliminates basis risk. This allows us to simplify the analysis and to focus on particular effects of an increase in risk under (μ, σ)-risk aversion when applying the concept of risk aversion elasticity.

2.1 (μ, σ)-risk aversion

(μ, σ)-risk aversion means that (i) preferences can be represented by a two-parameter function $U(\mu, \sigma)$ defined over mean μ and standard deviation σ of the underlying random variable and (ii) that the function U satisfies the following properties: $\partial U(\mu,\sigma)/\partial\mu = U_\mu > 0$, $\partial^2 U(\mu,\sigma)/\partial\mu^2 = U_{\mu\mu} \leq 0$, $\partial U(\mu,\sigma)/\partial\sigma = U_\sigma < 0$, $\sigma > 0$ and $U_\sigma(\mu,0) = 0$. We assume that partial derivatives $\partial^2 U(\mu,\sigma)/\partial\sigma^2$ and $\partial^2 U(\mu,\sigma)/\partial\mu\partial\sigma$ exist and that U is a strictly concave function. Hence, indifference curves are convex in (σ,μ)-space as often assumed in the literature (Eichberger and Harper, 1997; Eichner 2000).

Given (μ,σ)-risk aversion the hedging decision problem of the firm reads:

$$\max_h U(\mu_{\tilde{w}}, \sigma_{\tilde{w}}),$$

where $\tilde{w} = \tilde{p}(1-h)x + p_f hx$ denotes uncertain time 1 (incremental) wealth, with hedge ratio h. We set $\mu_{\tilde{w}} = E(\tilde{w})$ and $\sigma_{\tilde{w}} = \sqrt{E(\tilde{w} - E(\tilde{w}))^2} > 0$.

Before analyzing a change in risk and its effect upon the optimal hedge ratio h, let us introduce risk aversion elasticity. To simplify notation we drop subscript $\tilde{w}$.

Definition (Risk aversion elasticity). If $\sigma > 0$, then the elasticity of risk aversion with respect to the standard deviation is given by $\varepsilon_{S,\sigma} = -\frac{\partial S}{\partial\sigma}\frac{\sigma}{S}$, where $S = -U_\sigma/U_\mu > 0$.

For S is the marginal rate of substitution between μ and σ, we interpret S as a measure of risk aversion in (σ,μ)-space (Löffler, 1996; Lajeri and Nielsen, 2000). Risk aversion elasticity $\varepsilon_{S,\sigma}$ is – in absolute value – given by percentage change in risk aversion divided by percentage change in standard deviation. The elasticity measure allows for an intuitively appealing interpretation of risk effects upon hedging (or, risk bearing) behavior of the firm.

2.2 Risk changes and the hedge ratio

We model a change in price risk as follows: $\tilde{p}(\beta) = E\tilde{p} + \beta(\tilde{p} - E\tilde{p})$, where the random variable $\tilde{p}$ has unit standard deviation and $0 < \beta < 1$. Then, increasing β models an increase in price risk. Substituting $\tilde{p}(\beta)$ for the random variable $\tilde{p}$ of the hedging decision problem generates a relationship between the optimal hedge ratio and price risk measured by the standard deviation of $\tilde{p}(\beta)$. Now we are ready to claim the following

Proposition: Assume backwardation in the futures market. Then the firm's optimum hedge ratio will increase when price risk increases if and only if risk aversion elasticity is less than unity. With unit risk aversion elasticity price risk changes will not alter optimum hedging.

Proof. Expected wealth at time 1 and its standard deviation are given by

$$E(\tilde{w}) = (1-h)xE(\tilde{p}(\beta)) + p_f hx,$$

and

$$\sigma = (1-h)x\sigma_{\tilde{p}(\beta)},$$

respectively. Hence, the objective function becomes

$$U\Big((1-h)xE(\tilde{p}(\beta)) + p_f hx,\ (1-h)x\sigma_{\tilde{p}(\beta)})\Big).$$

By using risk aversion measure S and standard deviation σ the first order condition of the hedging decision problem becomes ($h(\beta) \neq 1$):

$$\left(p_f - E(\tilde{p}(\beta)) + \frac{\sigma S}{(1-h(\beta))}\right) U_\mu = 0,$$

which will be satisfied if and only if the term in brackets vanishes, since $U_\mu > 0$. With backwardation, i.e., $p_f < E(\tilde{p}(\beta))$, we obtain $0 < h(\beta) < 1$ for the relevant range of β. The implicit function theorem then gives

$$\begin{aligned} \text{sign}\left(\frac{dh(\beta)}{d\beta}\right) &= \text{sign}\left(\frac{1}{1-h(\beta)}\left\{\frac{\partial\sigma}{\partial\beta}S + \sigma\frac{\partial S}{\partial\sigma}\frac{\partial\sigma}{\partial\beta}\right\}\right) \\ &= \text{sign}\left(S + \sigma\frac{\partial S}{\partial\sigma}\right), \end{aligned}$$

since $1 - h(\beta) > 0$ and $\partial\sigma/\partial\beta > 0$. Applying the elasticity definition we obtain $\text{sign}[dh(\beta)/d\beta] = \text{sign}[1 - \varepsilon_{S,\sigma}]$. □

Our result is an application of Battermann et al., 2002, to the hedging decision of the firm. The two-parameter function $U(\mu,\sigma) = \mu/(\theta+\sigma)^2$, which is related to Roy's function (Roy, 1952), includes all three cases of our proposition, i.e., $\varepsilon_{S,\sigma} < (=)[>]\, 1$ if $\theta > (=)[<]\, 0$.

The figure below exhibits a graphical illustration of our result.

Let L_1 be the initial opportunity line on the relevant range as depicted in the figure: $L_1 = \{(\mu, \sigma) : \mu = \mu_{\tilde{w}},\ \sigma = \sigma_{\tilde{w}}, 0 < h(\beta_1) < 1$, given $\tilde{p}(\beta_1), \beta_1 > 0\}$. Optimal hedging satisfies the tangency condition (point A). The slope of the opportunity line equals the marginal rate of substitution, where $U(\mu, \sigma) = c_1$ denotes an indifference curve with utility level c_1. If standard deviation of $\tilde{p}(\beta)$ increases, then the new opportunity line $L_2 = \{(\mu, \sigma) : \mu = \mu_{\tilde{w}},\ \sigma = \sigma_{\tilde{w}},\ 0 < h(\beta_2) < 1$, given $\tilde{p}(\beta_2),\ \beta_2 > \beta_1\}$ becomes flatter. Hence utility level decreases ($c_2 < c_1$) and optimum policy moves from point A to point C.

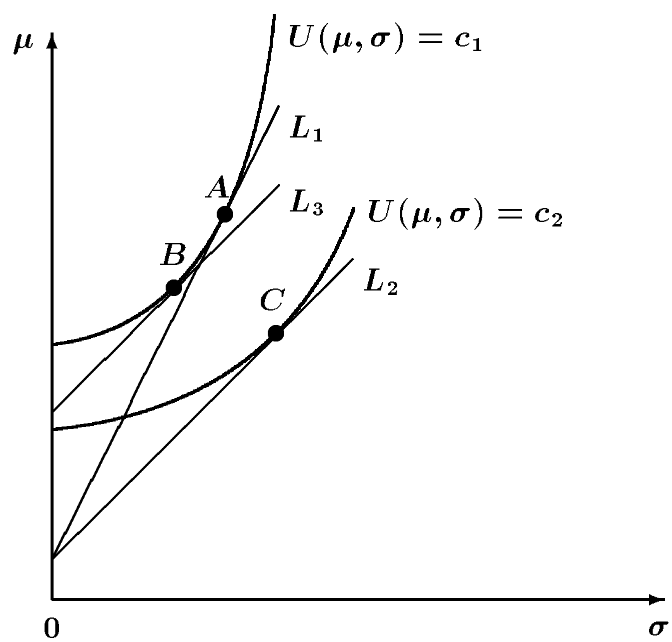

Figure: Substitution and income effect of risk

Now suppose that expected time 1 wealth increases for any level of risk such that the firm reaches the initial indifference curve $U(\mu, \sigma) = c_1$. The resulting opportunity line L_3 (a parallel shift of line L_2) leads to the new optimum point B. This shows the negative substitution effect (Davis, 1989).

Let us now reduce expected time 1 wealth for any level of risk. Changing line L_3 to line L_2 leads to the optimum point C. The move from A to B

provides the reason for the substitution effect and the move from B to C for the income effect. In the figure the income effect is dominated by the substitution effect which is saying that risk aversion elasticity is less than unity. If risk aversion elasticity is to be greater than unity, then the income effect dominates the substitution effect. In this case the optimum hedge ratio decreases, i.e. risk bearing by the firm increases, although price risk increases. In contrast to the existing hedging literature this is a remarkable simple characterization.

Note that our proposition can also be used under the expected utility hypothesis. The (μ, σ)-decision model is not in conflict with maximizing expected utility but has notably attractive properties. For example, it can be shown by the findings of Schneeweiß (1967), Sinn (1983), Meyer (1987), and Lajeri and Nielsen (2000) that the elasticity of risk aversion is always less than unity if Bernoulli-preferences display decreasing absolute risk aversion in the sense of Arrow and Pratt.

3. Concluding remarks

We have analyzed the need of revising optimum hedging policy of a firm when price volatility changes. We find that risk aversion elasticity determines whether or not a (μ, σ)-risk averse firm (or, a risk averse expected utility maximizing firm) decreases or increases its optimum hedge ratio when market prices become more volatile. We value our result as a remarkable simple characterization.

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