MEASURING VALUE AT RISK UNDER THE CONDITIONAL EDGEWORTH-SARGAN DISTRIBUTION.

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Abstract:

This paper introduces the Edgeworth-Sargan distribution on measuring Value-at-Risk of portfolios. The flexible parametric representation of this density makes it capable of improving the density fits (especially at the tails) and permits a straightforward method of percentile computation. Moreover, the time varying variance-covariance structure of the portfolio can be also estimated consistently to the Edgeworth-Sargan hypothesis since this density admits a straightforward multivariate representation. Estimates of portfolio's VaR evidence the underestimation of VaR measures under the normality assumption and also the more accurate VaR measures under the Edgeworth-Sargan distribution.

Keywords: Value at Risk, multivariate densities, Edgeworth-Sargan density, GARCH models.

JEL classification code: G10, G11, C13.

1. Introduction.

This paper focuses on measuring Value-at-Risk under the assumption of the Edgeworth-Sargan distribution – hereafter ES – as the data generating process. The rationale under such a distribution – Sargan (1976 and 1980) – lies in its ability to account for thicker tails than the normal, as well as for possible asymmetries as a result of the consideration of a general and flexible parameterisation. This flexibililiy is due to the density representation based on Edgeworth or Gram-Charlier expansions (Nishiyama and Robinson, 2000; Jondeau and Rockinger, 2001, respectively). From an empirical perspective, this distribution has been shown capable of accounting for salient empirical regularities of the histogram for most high frequency financial variables. This result was shown by Mauleon and Perote (2000) who tested the ES performance compared to other fitted densities like the Student's t.

Despite the Edgeworth-Sargan ability to represent the behaviour of most high frequency financial variables, this distribution has not been much used in finance and, as far as we know, this is its first application to measuring Value-at-Risk. Therefore, we propose a methodological approach to calculate the portfolio VaR by assuming that each portfolio's asset is conditionally ES distributed. Another interesting property of such distribution is that linear transformations of ES variables are also ES distributed, thus the linear portfolios of ES variables are also ES distributed. Moreover the variance and covariance matrices can be estimated consistently under such hypothesis since multivariate ES densities can also be provided. Finally, the traditional ARIMA or GARCH models can be used to model the conditional mean and variance. Actually, their simplest versions -AR(1) and GARCH(1,1)- are used in this article.

Next Section (2) describes the general model considered for calculating a portfolio's VaR, and Section 3 displays some empirical results. VaR for different portfolios are calculated by applying this methodological approach for a sample of eitghteen-years daily observations of stock indices and long run interest rates for major financial markets. Finally, conclusions are gathered in Section 4.

2. Portfolio model.

Let us consider that the random vector $R_t' = \left[r_{1t}, r_{2t}, \cdots, r_{Nt} \right]$ is distributed according to the joint density function $\Phi(\bullet)$, conditionally on the known information set Ω_{t-1}. Let us also assume that the conditional mean and the conditional variance-covariance matrix are denoted by μ_t and Σ_t respectively, i.e. $R_t \left| \Omega_{t-1} \right. \sim \Phi(\mu_t, \Sigma_t)$, where

$$E\left[R_t \left| \Omega_{t-1} \right. \right] = \mu_t = \begin{bmatrix} \mu_{1t} \\ \mu_{2t} \\ \vdots \\ \mu_{Nt} \end{bmatrix} \text{ and } E\left[(R_t - \mu_t)(R_t - \mu_t)' \left| \Omega_{t-1} \right. \right] = \Sigma_t = \begin{bmatrix} k_{1t}^2 & k_{12t} & \cdots & k_{1Nt} \\ k_{21t} & k_{2t}^2 & \cdots & k_{2Nt} \\ \vdots & \vdots & \vdots & \vdots \\ k_{1Nt} & k_{2Nt} & \cdots & k_{Nt}^2 \end{bmatrix}. \tag{2.1}$$

Therefore, the marginal density of each variable will be defined as

$$r_{it} \left| \Omega_{t-1} \right. \sim \phi_i(\mu_{it}, k_{it}^2), \ \forall i = 1,2,\ldots,N. \tag{2.2}$$

In particular we focus on two alternative distributions, the normal and the so called Edgeworth-Sargan distribution. The first one can be described by the following density,

$$G(R_t) = (2\pi)^{-N/2} \left| \Sigma_t^{-1/2} \right| \exp\left\{ -\frac{1}{2}(R_t - \mu_t)' \Sigma_t^{-1} (R_t - \mu_t) \right\} \tag{2.3}$$

whilst the second is just a generalisation of this density to the following family

$$F(R_t) = G(R_t) + \left[\prod_{i=1}^{N} g_i(v_{it}) \right] \left[\sum_{i=1}^{N} q_i(v_{it}) \right] \tag{2.4}$$

where $v_{it} = \dfrac{r_{it} - \mu_{it}}{k_{it}}$, $G(R_t)$ stands for the multivariate normal, $g_i(\bullet)$ stands for their marginal distributions (i.e. univariate normal distributions), and $q_i(\bullet) = \sum_{s=2}^{S_i} d_{si} H_s(\bullet)$ are the corresponding linear combination of Hermite polynomials.[1] These polynomials can be defined as those $H_s(v_{it})$ satisfying

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[1] Note that the polynomial truncation (i.e. the choice of S_i) depend on the accuracy of the fitted density compared to the empirical histogram of the data (for example). Nevertheless the

$$\frac{d^j g(v_{it})}{dv_{it}^j} = (-1)^j g(v_{it}) H_s(v_{it}) , \tag{2.5}$$

where $g(\bullet)$ stands for a standard normal density, i.e. $N(0,1)$. A straightforward calculation of Hermite polynomials[2] is given by Kendall and Stuart (1977):

$$H_j(v_{it}) = v_{it}^j - \frac{j(j-1)}{2} v_{ij}^{j-2} + \frac{j(j-1)(j-2)(j-3)}{2^2 2!} v_{ij}^{j-4} - \frac{j(j-1)(j-2)(j-3)(j-4)(j-5)}{2^3 3!} v_{it}^{j-6} + \cdots \tag{2.6}$$

It must be noted that the expression 2.4 given for the multivariate Edgeworth-Sargan distribution was developed by Mauleón and Perote (1999). These authors showed that the marginal densities of this distribution are also univariate Edgeworth-Sargan distributions, i.e.

$$f_i(r_{it}) = \frac{1}{k_{it}} g(v_{it}) \{1 + q_i(v_{it})\}, \tag{2.7}$$

where $g(\bullet)$ stands for the standard normal density. Moreover, it is easy to check that both the multivariate and univariate ES integrate up to one, are positive and symmetric functions and have thicker tails than the normal distribution (provided that $d_{si} > 0$ for even values of i and $d_{si} = 0$ for odd values of i).[3] Even more, as linear combinations of ES variables are also ES

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improvements in terms of the quality of data fits (and VaR measures) are obtained at the cost of an increasing complexity of the distribution and therefore require the implementation of "efficient" parameter estimation algorithms.

[2] In particular, the polynomials used for the empirical examples presented in section 3 are the following: $H_2(v_{it}) = v_{it}^2 - 1$, $H_4(v_{it}) = v_{it}^4 - 6v_{it}^2 + 3$, $H_6(v_{it}) = v_{it}^6 - 15v_{it}^4 + 45v_{it}^2 - 15$ and $H_8(v_{it}) = v_{it}^8 - 28v_{it}^6 + 210v_{it}^4 - 420v_{it}^2 + 105$.

[3] Although one of the advantages of the use of ES distribution is the fact that this distribution can represent asymmetries (by means of the odd parameters) we decided to impose the symmetric case after having tested these hypotheses. Our results shown that it is not possible to reject the non significance of the odd parameters (this result does not support previous evidence found by authors such as Theodossiou, 1998).

distributed, the portfolio P_t defined[4] in 2.8 is clearly distributed as indicated in 2.9:

$$P_t = W_t' R_t = \sum_{i=1}^{N} w_{i,t-1} r_{it} \tag{2.8}$$

$$f_p(P_t) = \frac{1}{k_{pt}} g\left(\frac{P_t - \mu_{pt}}{k_{pt}}\right)\left\{1 + q_p\left(\frac{P_t - \mu_{pt}}{k_{pt}}\right)\right\}. \tag{2.9}$$

Note that $q_p(\bullet) = \sum_{s=2}^{S_p} d_{sp} H_s(\bullet)$ and portfolio's conditional mean and variance are displayed in 2.10 and 2.11 respectively.

$$\mu_{pt} = W_t' \mu_t = \sum_{i=1}^{N} w_{i,t-1} \mu_{it} \tag{2.10}$$

$$k_{pt}^2 = W_{t-1}' \Sigma_t W_{t-1} = \sum_{i=1}^{N} w_{i,t-1}^2 k_{it}^2 + \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} w_{i,t-1} w_{j,t-1} k_{ijt} \ . \tag{2.11}$$

The portfolio Value-at-Risk describes the worst expected loss of the portfolio in some temporal horizon and at some confidence level. Hence the computation of VaR requires not only calculating the conditional portfolio variance but also the whole portfolio conditional distribution. In particular, once the distribution percentile $\phi(\alpha)$ has been computed, the portfolio VaR at a confidence level α can be defined as follows:

$$VaR(\alpha) = k_{pt}\phi(\alpha). \tag{2.12}$$

Therefore the search of accurate VaR measures involves the estimation of the right portfolio distribution and the computation of the corresponding percentile. That is the reason why we propose the use of a more flexible distribution[5] to capture the underlying density of the

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[4] $W_t' = \begin{bmatrix} w_{1t} & w_{2t} & \cdots & w_{Nt} \end{bmatrix}$ such that $0 \le w_{i,t-1} \le 1 \ \forall i = 1,2,...,N$ and $\sum_{i=1}^{N} w_{i,t-1} = 1$.

[5] The ES is not only more flexible than the Normal, but also than other distributions that attempt to account for extreme values probability (see Mauleón and Perote, 2000).

portfolio by means of a larger number of parameters[6]. Moreover, this distribution is easy to integrate and, therefore, the percentile of the distribution can be computed by considering the following property:

$$\alpha = \Pr ob\left[x \le \phi(\alpha)\right] = \int_{-\infty}^{\phi(\alpha)} f(x)dx = \int_{-\infty}^{\phi(\alpha)} g(x)dx - g(\phi(\alpha))\sum_{s=2}^{S_p} d_s H_{s-1}(\phi(\alpha)) \,. \tag{2.13}$$

Finally, to account for conditional mean and conditional variance we assume some of the most widespread models in the financial literature: the AR(1) and GARCH(1,1) processes. Accordingly, we suppose that

$$\mu_{it} = \phi_{0i} + \phi_{1i} r_{it-1} \text{ and} \tag{2.14}$$

$$k_{it}^2 = \alpha_{0i} + \alpha_{1i} k_{i,t-1}^2 + \alpha_{2i} \varepsilon_{i,t-1}^2 \,, \tag{2.15}$$

where $\varepsilon_{it} = r_{it} - \mu_{it}$, $|\phi_{1i}| < 1$, $\alpha_{0i} > 0$, $0 \le \alpha_{si} \le 1$ $\forall s = 1,2$ and $\alpha_{1i} + \alpha_{2i} < 1$, $\forall i = 1,2,...,N$. Moreover, we assume constant correlation coefficients (ρ_{ijt}) and thus conditional covariances can be written as

$$k_{ijt} = \rho_{ij} k_{it} k_{jt} \tag{2.16}$$

where $-1 \le \rho_{ij} \le 1$ $\forall i = 1,2,...,N$ and $\forall j = 1,2,...,N$.

3. Empirical results.

The model shown in previous section was estimated by maximum likelihood[7] assuming both the normal and the ES distribution and also accounting for conditional heteroskedasticity. In particular, Tables 1 and 2 display the parameter estimates for the joint conditional

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[6] Note that there exists a direct relation among the ES parameters and its moments, which implies that this method of VAR estimation takes into account higher order moments as well as the variance.

distributions of different portfolios[8] of stock indices (Dow Jones and FTSE) and long run interest rates. In addition, we show the estimates for the portfolio parameters (including the portfolio conditional variance and log-likelihood values), the portfolio's VaR and the differences in the VaRs obtained under the ES and the Normal. Different confidence levels (0.05, 0.025, and 0.01) and weights (0.1, 0.5 and 0.9) are considered for the estimation.[9]

According to the obtained log likelihood values,[10] we conclude that the multivariate conditional ES distribution represents the joint behaviour of portfolio variables more accurately than the multivariate conditional normal. In consequence, portfolio's VaR is clearly misleading[11] when accounting only for the first and second moments (i.e. under the normal distribution). In this sense the differences are larger in high volatility scenarios and for low confidence levels. Another feature that must be noted is the non-stationarity of the GARCH processes, which is also consistent with the RiskMetriks methodology. As is well known, this persistence of the conditional variances captures part of the unconditional kurtosis (Engle and Bollerslev, 1986) and, thus, the VaR differences are not as large as those that would be found if unconditional normal and ES distributions were estimated.

Finally, it is also noteworthy that this approach allow to analyse the sensitivity of the VaR to changes in the portfolio weights. For example it is clear from Table 1 that portfolio

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[7] Notice that the computational problems derived from the complexity of the model can be solved by adequately selecting the initial values for the optimisation algorithms. Initial values are based on the implicit relation between ES parameters and moments.

[8] For the sake of simplicity, we only consider two asset portfolios and three different weights: 0.1, 0.5 and 0.9.

[9] For the sake of brevity, the results obtained for some other portfolios were omitted. They are available under request.

[10] As the multivariate normal is nested on the multivariate ES, the LR test can be straightforward implemented to test the Normal versuse the ES.

[11] This evidence is consistent with other authors like Lucas (2000) or Danielsson and de Vries (2000) when comparing normal-VaRs to those obtained under a Student's t distribution or a semi-parametric method.

VaR increases as the weight of the long run interest descends. Therefore, the estimates of different portfolios can help risk managers to choose the strategies that minimise VaR at any period depending on weights and risk aversion.

4. Conclusions.

The ES density has been shown capable of fitting financial univariate densities more accurately than other popular densities used in finance like the Student's t (see Mauleón and Perote, 2000). This distribution can also be generalised to a multivariate context and thus used to estimate the whole distribution of portfolio variables. This article proposes to calculate portfolio's VaR consistently with the ES distributional hypothesis and applies it to the estimation of portfolio's VaR at different confidence levels. For the sake of comparison, the analyses are carried out under both the traditional Normal and the ES distribution. The main conclusions of the study can be summarised as follows:

1.- The VaR methodologies should incorporate the non-normality of most high frequency financial variables. The ES distribution is able to account for the main empirical features of most high frequency data and provides a flexible and simple parametric representation of the underlying density. Moreover, this distribution generalise the Normal and, therefore, its implementation and the comparison of both specifications are straightforward.

2.- The variance and covariance matrix of portfolio variables may be estimated consistently with the ES hypothesis, since this density can also be generalised to a multivariate context. Additionally, different time varying variance hypotheses, such as a GARCH(1,1), can be also implemented.

3.- The portfolio VaR computed under the more reasonable ES distribution seemed to be bigger than the traditional VaR (under the normal hypothesis). This VaR understatement is especially highlighted for low confidence levels and high volatility scenarios (risk adverse investors). Therefore, the flexible representation of the ES distribution can help risk managers to improve

VaR measures by considering not only the conditional variance, but also other moments (parameters).

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TABLE 1: Dow Jones index and long run interest rates in United States (4/1/83 – 13/11/87).

Joint Distributions and portfolio's VaR						
Weight[1]	$w_1 = .1$		$w_1 = .5$		$w_1 = .9$	
Distribution	NORMAL	ES	NORMAL	ES	NORMAL	ES
ϕ_{01}	$.23\times10^{-3}$	$.23\times10^{-3}$	$.23\times10^{-3}$	$.23\times10^{-3}$	$.23\times10^{-3}$	$.23\times10^{-3}$
ϕ_{11}	$.0215^{*}$	$.0215^{*}$	$.0215^{*}$	$.0215^{*}$	$.0215^{*}$	$.0215^{*}$
α_{01}	$.28\times10^{-5}$	$.25\times10^{-5}$	$.28\times10^{-5}$	$.25\times10^{-5}$	$.28\times10^{-5}$	$.25\times10^{-5}$
α_{11}	.845	.858	.845	.858	.845	.858
α_{21}	.149	.156	.149	.156	.149	.156
D_{41}		.677		.677		.677
D_{81}		.03		.03		.03
ϕ_{02}	$.16\times10^{-3*}$	$.16\times10^{-3*}$	$.16\times10^{-3*}$	$.16\times10^{-3*}$	$.16\times10^{-3*}$	$.16\times10^{-3*}$
ϕ_{12}	$.0153^{*}$	$.0153^{*}$	$.0153^{*}$	$.0153^{*}$	$.0153^{*}$	$.0153^{*}$
α_{02}	$.78\times10^{-6*}$	$.47\times10^{-7}$	$.78\times10^{-6*}$	$.47\times10^{-7}$	$.78\times10^{-6*}$	$.47\times10^{-7}$
α_{12}	.957	.979	.957	.979	.957	.979
α_{22}	.043	.034	.043	.034	.043	.034
D_{42}		.189		.189		.189
D_{82}		.0046		.0046		.0046
ρ_{12}	–.287	–.645	–.287	–.645	–.287	–.645
Log L	8,795	9,576	8,795	9,576	8,795	9,576
PORTFOLIO						
D_{2p}		.168		.318		$-.037^{*}$
D_{4p}		.107		.108		.0568
D_{6p}		.018		.0075		.0061
D_{8p}		.0017		$-.0009^{*}$		.0014
Log L		4,233		4,624		4,194
σ_p (.05)	.00826	.00672	.01205	.01066	.0223	.0228
VaR (.05)	.01363	.01281	.02312	.01989	.0367	.0376
% (.05)		–6.05		16.24		2.43
VaR (.025)	.01611	.01586	.0235	.02873	.0434	.0458
% (.025)		–1.52		22.22		5.53
VaR (.01)	.01924	.02043	.02808	.03429	.0519	.0558
% (.01)		6.13		22.02		7.67

[1] Weight for the Dow Jones index * The parameter is not significant at 5% confidence level.

TABLE 2: Dow Jones and FTSE indices (1/1/70 – 13/11/87).

Joint Distribution and portfolio's VaR						
Weight[1]	$w_1 = .1$		$w_1 = .5$		$w_1 = .9$	
Distribution	NORMAL	ES	NORMAL	ES	NORMAL	ES
ϕ_{01}	$.14\times10^{-3}$	$.14\times10^{-3}$	$.14\times10^{-3}$	$.14\times10^{-3}$	$.14\times10^{-3}$	$.14\times10^{-3}$
ϕ_{11}	.025	.025	.025	.025	.025	.025
α_{01}	$.53\times10^{-6}$	$.84\times10^{-6}$	$.53\times10^{-6}$	$.84\times10^{-6}$	$.53\times10^{-6}$	$.84\times10^{-6}$
α_{11}	.968	.959	.968	.959	.968	.959
α_{21}	.037	.046	.037	.046	.037	.046
D_{41}		.385		.385		.385
D_{81}		.0211		.0211		.0211
ϕ_{02}	$.39\times10^{-3}$	$.39\times10^{-3}$	$.39\times10^{-3}$	$.39\times10^{-3}$	$.39\times10^{-3}$	$.39\times10^{-3}$
ϕ_{12}	.093	.093	.093	.093	.093	.093
α_{02}	$.24\times10^{-5}$	$.18\times10^{-5}$	$.24\times10^{-5}$	$.18\times10^{-5}$	$.24\times10^{-5}$	$.18\times10^{-5}$
α_{12}	.886	.876	.886	.876	.886	.876
α_{22}	.095	.112	.095	.112	.095	.112
D_{42}		.0455		.0455		.0455
D_{82}		.0034		.0034		.0034
ρ_{12}	.119	.2586	.119	.2586	.119	.2586
Log L	33,043	33,958	33,043	33,958	33,043	33,958
PORTFOLIO						
d_{2p}		$-.014^{*}$		$-.084$		$-.086$
d_{4p}		.0176		.017		.0079
d_{6p}		.0025		$.0002^{*}$		$-.0048$
d_{8p}		.0003		.0003		$.00006^{*}$
Log L		15,303		16,356		15,712
σ_p (.05)	.00719	.00737	.00952	.01063	.01504	.01648
VaR (.05)	.01187	.01216	.01571	.01754	.02482	.02719
% (.05)		2.42		11.66		9.57
VaR (.025)	.01403	.01445	.01856	.02083	.02933	.03231
% (.025)		2.95		12.21		1.13
VaR (.01)	.01677	.01724	.02476	.02476	.03505	.03841
% (.01)		2.86		11.64		9.57

[1] Weight for the Dow Jones index * The parameter is not significant at 5% confidence level.