# Optimal Portfolios and Heston's Stochastic Volatility Model

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ABSTRACT: Given an investor maximizing utility from terminal wealth with respect to a power utility function, we present a verification result for portfolio problems with stochastic volatility. Applying this result, we solve the portfolio problem for Heston's stochastic volatility model.

KEYWORDS: optimal portfolios, stochastic volatility, Heston model

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# 1 Introduction

Starting with the seminal papers by Merton (1969, 1971) there has been a lot of research on continuous-time portfolio optimization. Whereas Merton considers a portfolio problem with a constant investment opportunity set, several authors look at portfolio problems with different variants of a stochastic opportunity set such as stochastic interest rates or stochastic volatility. For the case of stochastic interest rates the reader is referred to Korn/Kraft (2001) and the references therein.

Given an investor maximizing utility from terminal wealth, Zariphopoulou (2001) considers a portfolio problem where the investor can put her money into a stock and a money market account. The stock is driven by a one-dimensional geometric Brownian motion with parameters depending on a one-dimensional geometric Brownian state process. The inherent Brownian motions may be not perfectly correlated so that incomplete market situations are covered. This setup is able to support portfolio problems with stochastic volatility. Zariphopoulou (2001) derives Feynman-Kac representations of the candidates for the value function and for the optimal portfolio process. By candidate for the value function we mean the process which solves the Hamilton-Jacobi-Bellman equation (if any). The corresponding portfolio process is said to be the candidate for the optimal portfolio process. Let us stress that in general these candidates need not be the value function and the optimal portfolio process. Applying viscosity solution techniques, Zariphopoulou (2001) is able to prove optimality of the candidates under some assumptions on the coefficients of the stochastic differential equations of the stock and the state process. More precisely, she assumes that linear growth as well as Lipschitz conditions are satisfied and that the market price of risk is bounded (Conditions (3.20), (3.21), and (3.22) in her paper). However, the first two assumptions exclude state processes which only meet the conditions of Yamada and Watanabe.[1] Probably the most prominent (and practically relevant) example which is thus not covered by her verification result is a portfolio problem with Heston's stochastic volatility model. Besides, the assumption of a bounded market price of risk does not cover portfolio problems, where the market price of risk equals some constant multiplied by some power of the volatility process. Such a specification was first proposed by Merton (1980) and is used in a couple of papers such as Bakshi/Chao/Chen (1997) and Bates (2000). Therefore, in this paper we prove a verification result which does not exclude such situations. Then we show that given the canonical formulation of Heston's volatility model with a market price of risk linear in the volatility process the candidate for the value function is only well-defined under some specific condition on the parameters of the model. Further, we are able to prove that this condition is also sufficient for the candidate to be indeed the value function. We want to stress that both a square-root state process and an unbounded market price of risk are not covered by Zariphopoulou's verification result. As however Heston's stochastic volatility model is something like an industry standard for option pricing, we believe that it is worth to solve a portfolio problem in this setting.

Let us now shortly discuss some other papers on portfolio problems with stochastic volatility including Chacko/Viceira (2002), Liu (2001a, 2001b), Pham (2002), and Fleming/Hernandez-H. (2003). Liu (2001a) considers Heston's model and comes up with the same candidate for the optimal portfolio strategy as we do for Heston's

[1] See e.g. Karatzas/Shreve (1991), p. 291.

model. The main problem with his results is that Korn/Kraft (2002) were able to construct examples showing that under some parametrizations of the model there exist infinitely many *bounded* portfolio strategies which lead to an infinite wealth. As Liu (2001a) does not present a verification result, it remains open under which conditions the candidate for the optimal portfolio strategy is indeed the unique optimal solution of the problem. Liu (2001b) treats the portfolio problem with a more general specification of the risk premium, but again he does not present a valid verification result. In contrast to Liu (2001a, b), Chacko/Viceira (2002) consider a portfolio problem with a different specification of the market price of risk. Besides, they use a slightly different stochastic volatility model and they also look at problems with intermediate consumption. However, they are only able to compute optimal portfolio strategies in special cases of their model and do not provide a verification result. In contrast to these papers, Pham (2002) and Fleming/Hernandez-H. (2003) derive explicit verification results proving that their portfolio strategies are indeed optimal. Given an investor maximizing power utility from terminal wealth, Pham (2002) considers a multidimensional model for securities with stochastic volatilities. However, he assumes that the coefficients of his volatility process satisfy a certain Lipschitz condition which excludes Heston's model (Condition (H1) in his paper). Fleming/Hernandez-H. (2003) consider an investor maximizing her lifetime consumption and assume the volatility of the assets to be a function $\sigma(\cdot)$ depending on a state process with constant volatility. Further, it is assumed that $\sigma(\cdot)$ is bounded away from zero as well as bounded from above and that the first derivative $\sigma'(\cdot)$ is bounded (Assumption A in their paper). These assumptions are not satisfied in important examples such as Heston's model.

To summarize, the contributions of this paper are the following:

- We prove a verification result which covers Heston's stochastic volatility model with an unbounded market price of risk.
- Given an (unbounded) market price of risk which equals a linear function of the volatility process, we derive a condition on the parameters of the model ensuring well-defined candidates in Heston's framework.
- We are able to show that this condition also ensures that the candidate for the value function and the optimal portfolio process are indeed the value function and the optimal portfolio process. The fact that such a condition has to be introduced in order to get a meaningful solution fundamentally distinguishes the results in this paper from Zariphopoulou's results.

The remainder of this paper is structured as follows: In Section 2 we introduce a Brownian framework for stochastic volatility which includes Heston's model as a special case. In Section 3 we shortly summarize Zariphopoulou's representation result concerning the candidate for the value function and the candidate for the optimal portfolio process. A verification result for portfolio problems with stochastic volatility which covers Heston's model is the subject of Section 4. Then, in Section 5 we consider Heston's model in detail and apply our verification result. Further, we look at different but related specifications of the market price of risk and of the volatility of the stock.

## 2 The Portfolio Problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space. On this space two correlated Brownian motions $\hat{W}_1$, $\hat{W}_2$ are given with $< \hat{W}_1, \hat{W}_2 >_t = \rho t$, $\rho \in [-1, 1]$. Additionally, $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the corresponding Brownian filtration. We consider an investor maximizing utility from terminal wealth at time $T$ with respect to a power utility function $U(x) = \frac{1}{\gamma} x^\gamma$, $x \geq 0$. We concentrate on the case $\gamma \in (0, 1)$ because it allows us to present the main ideas. The investment opportunities include a money market account with the dynamics

$$dM(t) = M(t) r dt,$$

$M(0) = 1$, where for simplicity the short rate $r$ is assumed to be constant. Additionally, the investor can put her money into a stock with the dynamics governed by the stochastic differential equation (SDE)

$$dS(t) = S(t)\Big[(r + \lambda(t))dt + \nu(t)\, d\hat{W}_2(t)\Big], \tag{1}$$

where it is assumed that $\lambda$ and $\nu$ are functions of time $t$ and the state process

$$dz(t) = \chi(t)dt + \delta(t) d\hat{W}_1(t). \tag{2}$$

Again we assume that $\chi$ and $\delta$ are functions of $t$ and $z$, i.e. $\chi(t) = \chi_f(t, z(t))$ and $\delta(t) = \delta_f(t, z(t))$ with real-valued functions $\chi_f$ and $\delta_f$. For instance, the specifications

$$\nu(t) = \sqrt{z(t)}, \quad \chi(t) = \kappa(\theta - z(t)),\ \kappa, \theta > 0, \quad \text{and} \quad \delta(t) = \sigma\sqrt{z(t)},\ \sigma > 0, \tag{3}$$

lead to Heston's model. The wealth equation for this problem reads as

$$dX^\pi(t) = X^\pi(t)\Big[\Big(r + \lambda(t)\pi(t)\Big)dt + \pi(t)\nu(t) d\hat{W}_2(t)\Big], \tag{4}$$

$X^\pi(0) = x_0 > 0$, where $\pi(t)$ denotes the proportion of wealth invested in the stock.[2] It is assumed that all coefficients of the above SDEs are progressively measurable with respect to the Brownian filtration $\{\mathcal{F}_t\}_t$ and that the SDEs have unique solutions. For (1) and (4) the latter requirement is met if[3]

$$\int_0^T |\lambda(s)| + \nu^2(s)\, ds \quad < \quad +\infty \ \ a.s. \ \text{ and} \tag{5}$$

$$\int_0^T |\lambda(s)\pi(s)| + \pi^2(s)\nu^2(s)\, ds \quad < \quad +\infty \ \ a.s. \tag{6}$$

For (2) we get uniqueness if - as assumed in the verification result of Zariphopoulou (2001) - global Lipschitz conditions

$$|\varpi(z', t) - \varpi(z'', t)| = K|z' - z''| \tag{7}$$

and linear growth conditions

$$\varpi^2(z', t) = \varpi(1 + z'^2) \tag{8}$$

on the coefficients are satisfied, where $z'$, $z'' \in [0, \infty)$, $t \in [0, T]$, $K$ is a positive constant, and $\varpi$ stands for $\chi_f$ and $\delta_f$ (Conditions (3.20) and (3.21) in her paper).

[2] For notational convenience, we sometimes omit the superindex $\pi$ of $X$.

[3] See e.g. Korn/Korn (2001), p. 54.

However, these assumptions exclude square-root processes such as the volatility process of Heston's model because $\delta$ in (3) does not meet condition (7). As mentioned in the introduction, the conditions of Yamada and Watanabe ensure existence and uniqueness for square-root processes and therefore we do not restrict our considerations to (7) and (8).[4] This is possible because the proof of our verification result does not explicitly require assumptions (7) and (8), but we only need the existence and uniqueness of solutions of the above SDEs.

The optimization problem of the investor reads as

$$\max_{\pi(\cdot)} \mathrm{E}\Big(\tfrac{1}{\gamma} X^\pi(T)^\gamma\Big)$$

with the corresponding value function

$$V(t,x,z) = \max_{\pi(\cdot)} \mathrm{E}^{t,x,z}\Big(\tfrac{1}{\gamma} X^\pi(T)^\gamma\Big).$$

In the sequel we mostly work with independent Brownian motions $W_1$ and $W_2$ given by $\hat{W}_1 = W_1$ and $\hat{W}_2 = \rho W_1 + \sqrt{1-\rho^2}\, W_2$.

# 3 The Representation Result by Zariphopoulou

In this section we summarize Zariphopoulou's representation result for the candidate of the value function. We want to stress that Zariphopoulou (2001) only assumes the above discussed Lipschitz and growth conditions (7) and (8) to prove her verification theorem and not to derive her representation result for the value function.

We face a two-dimensional control problem with state process $(X, z)$. The Hamilton-Jacobi-Bellman equation which has to be satisfied by the candidate $G$ for the value function $V$ reads as[5]

$$\sup_\pi \Big\{ \underbrace{G_t + x(r+\lambda\pi)G_x + \chi G_z + 0.5x^2\nu^2\pi^2 G_{xx} + \delta x \nu \pi \rho G_{xz} + 0.5\delta^2 G_{zz}}_{=: \mathcal{A}^\pi G} \Big\} = 0 \tag{9}$$

with the terminal condition $G(T,x,z) = \frac{1}{\gamma}x^\gamma$. By Zariphopoulou (2001), the candidate $G$ of the value function possesses the following representation

$$G(t,x,z) = \tfrac{1}{\gamma}x^\gamma \cdot \Big(f(t,z)\Big)^c \qquad \text{with } f(T,z) = 1 \text{ for all } z$$

and $c = \frac{1-\gamma}{1-\gamma+\rho^2\gamma}$. Let $\tilde{r} := -\frac{\gamma}{c}\Big(r + 0.5\frac{1}{1-\gamma}\frac{\lambda^2}{\nu^2}\Big)$ and $\tilde{\chi} := \Big(\chi + \frac{\gamma}{1-\gamma}\frac{\lambda}{\nu}\rho\delta\Big)$. Then the function $f$ has the Feynman-Kac representation

$$f(t,z) = \tilde{\mathrm{E}}^{t,z}\bigg(e^{-\int_t^T \tilde{r}(s)\, ds}\bigg) \tag{10}$$

with

$$dz(t) = \tilde{\chi}(t)dt + \delta(t)d\tilde{W}_1(t),$$

[4] To prove her verification result, Zariphopoulou (2001) additionally assumes that condition (7) is also satisfied by the coefficients of (1). In general this is a stronger requirement then our condition (5).

[5] For notational convenience, we often omit the functional dependencies with respect to $t$, $x$ and $z$.

where the measure is changed to $\tilde{P}$ which is defined by the Girsanov density

$$Z(t) := \frac{d\tilde{P}}{dP}\bigg|_{\mathcal{F}_t} = \exp\Big( -0.5(\tfrac{\gamma}{1-\gamma})^2\rho^2 \int_0^t \zeta^2(s)\,ds + \tfrac{\gamma}{1-\gamma}\rho \int_0^t \zeta(s)\,dW_1(s)\Big)$$

with the market price of risk $\zeta(t) := \lambda(t)/\nu(t)$. The corresponding expectation is denoted by $\tilde{\mathrm{E}}$ and the process

$$\tilde{W}_1(t) = W_1(t) - \tfrac{\gamma}{1-\gamma}\rho \int_0^t \zeta(s)\,ds$$

is a Brownian motion under the measure $\tilde{P}$. Further, the candidate for the optimal portfolio strategy can be written as

$$\pi^*(t) = \frac{1}{1-\gamma}\frac{\lambda(t)}{\nu^2(t)} + \frac{1}{1-\gamma} c\rho \frac{\delta(t)}{\nu(t)} \frac{f_z(t,z(t))}{f(t,z(t))}.$$

We want to stress that these results are *only meaningful if* $Z$ is a density and $f$ is well-defined.

# 4 A Verification Result

As just mentioned, the representation results in Section 3 need not always be meaningful. However, if $Z$ is a density and $f$ is in $C^{1,2}[0,T]$, we can give a suitable verification result. To formulate this result we need some definitions. Without loss of generality we assume that the range of $z$ equals $[0,\infty)$.

**Definition 4.1 (Admissible Portfolio Strategy)** *A portfolio strategy $\pi$ is said to be admissible if the following conditions are satisfied:*

*(i) $\pi$ is progressively measurable,*

*(ii) for all initial conditions $(t_0, x_0, z_0) \in [0,T] \times (0,\infty)^2$ the wealth process $X^\pi$ with $X^\pi(t_0) = x_0$ has a pathwise unique solution $\{X^\pi(t)\}_{t\in[t_0,T]}$,*

*(iii) $\mathrm{E}^{t_0,x_0,z_0}\Big(\Big[\frac{1}{\gamma}X^\pi(T)^\gamma\Big]^-\Big) < +\infty$,*

*(iv) $X^\pi \geq 0$.*

*We denote the set of admissible strategies by $\mathcal{A}$. Besides, $\mathcal{A}_2$ denotes the subset of all admissible strategies $\pi$ belonging to $L^2[0,T]$, i.e.*

$$\mathrm{E}\Big(\int_0^T \pi^2(s)\,ds\Big) < \infty.$$

By assumption (6), the wealth equation (4) has the unique solution

$$X(t) = X(0)\cdot \exp\Big(\int_0^t r + \lambda(s)\pi(s) - 0.5\pi^2(s)\nu^2(s)\,ds + \int_0^t \pi(s)\nu(s)\,d\hat{W}_2(s)\Big)$$

and thus (ii) and (iv) are met. Further, for $\gamma > 0$ the condition (iii) is satisfied by all means.

**Definition 4.2 (Property U)** *Assume that $Z$ is well-defined and that we have $f \in C^{1,2}[0,T]$. Let $\pi \in \mathcal{A}$. If for all sequences of stopping times $\{\theta_p\}_{p\in\mathbb{N}}$ with $0 \leq \theta_p \leq T$ the sequence $\{G(\theta_p, X^\pi(\theta_p), z(\theta_p))\}$ is uniformly integrable, then we say that $\pi$ has property U.*

In Section 2 we assumed that $\nu$ and $\delta$ are functions of $t$ and $z(t)$, i.e. $\nu(t) = \nu_f(t, z(t))$ and $\delta(t) = \delta_f(t, z(t))$ with real-valued functions $\nu_f$ and $\delta_f$. Now we make the additional assumption that for all bounded sets $I \subset [0,T] \times [0,\infty)$ there exists some constant $K$ such that

$$|\nu_f(t,z)| + |\delta_f(t,z)| \leq K \quad \text{for all } (t,z) \in I. \tag{11}$$

Note that this condition is satisfied if $\nu_f$ and $\delta_f$ are continuous functions. Then we can prove the following proposition:

**Proposition 4.1 (Verification Result)** *Assume that $Z$ is well-defined and that $f \in C^{1,2}[0,T]$. Then we obtain*

$$\mathrm{E}^{t_0,x_0,z_0}\left(\tfrac{1}{\gamma}X^{\pi}(T)^{\gamma}\right) \leq G(t_0,x_0,z_0) \quad \text{for all } \pi \in \mathcal{A}. \tag{12}$$

*Now assume $\pi^* \in \mathcal{A}_2$. If (11) holds and $\pi^*$ has property U, then we get*

$$\mathrm{E}^{t_0,x_0,z_0}\left(\tfrac{1}{\gamma}X^{\pi^*}(T)^{\gamma}\right) = G(t_0,x_0,z_0). \tag{13}$$

**Proof**. See Appendix.

Conditions (12) and (13) ensure that $G$ is the value function of the problem and $\pi^*$ is the optimal portfolio strategy. In contrast to Zariphopoulou (2001) we have to make the assumption $f \in C^{1,2}[0,T]$, while she was able to show this but under strong assumptions (Lipschitz and growth conditions as well as bounded market price of risk). In the following section we will see that in a Heston model with unbounded market price of risk $f$ is not even well-defined in general. Therefore, one cannot expect a result as in Zariphopoulou (2001) to hold.

# 5 A Portfolio Problem within Heston's Setting

In a Heston model with

$$\begin{aligned} \chi(t) &= \kappa(\theta - z(t)), \qquad \kappa,\theta > 0, \\ \delta(t) &= \sigma\sqrt{z(t)}, \qquad \sigma > 0, \\ \lambda(t) &= \bar{\lambda}\cdot z(t), \qquad \bar{\lambda} \in \mathbb{R}, \\ \nu(t) &= \sqrt{z(t)} \end{aligned} \tag{14}$$

it is not obvious if $Z$ and $f$ are well-defined because the market price of risk $\zeta(t) = \lambda(t)/\nu(t) = \bar{\lambda}\sqrt{z(t)}$ is unbounded. Thus, we first derive a condition under which this is valid. Then we apply our verification result.

## 5.1 Well-defined Candidates in Heston's Model

The process $z$ is given by

$$dz(t) = \kappa(\theta - z(t))dt + \sigma\sqrt{z(t)}dW_1(t) \tag{15}$$

under $P$ and by

$$\begin{aligned} dz(t) &= \left(\kappa(\theta - z(t)) + \tfrac{\gamma}{1-\gamma}\rho\bar{\lambda}\sigma z(t)\right)dt + \sigma\sqrt{z(t)}d\tilde{W}_1(t) \\ &= \tilde{\kappa}\left(\tfrac{\kappa\theta}{\tilde{\kappa}} - z(t)\right)dt + \sigma\sqrt{z(t)}d\tilde{W}_1(t) \end{aligned} \tag{16}$$

with $\tilde{\kappa} = \kappa - \frac{\gamma}{1-\gamma}\rho\bar{\lambda}\sigma$ under $\tilde{P}$. Note that for $\tilde{\kappa} = 0$ the drift simply equals $\kappa\theta$. Our first task is to analyze under which conditions $Z$ is a density and $f$ is well-defined. For this reason, we apply the following variant of a theorem by Pitman and Yor.

**Proposition 5.1** *Consider the process (15) and let*

$$\varphi(t,T,z) := \mathrm{E}\Big( \exp\Big( -\alpha z(T) - \beta \int_t^T z(s)\,ds \Big) \Big| z(t) = z \Big) \tag{17}$$

*be the characteristic function of* $\Big(z(T), \int_t^T z(s)\,ds\Big)$. *Then* $\varphi$ *is well-defined if*

$$\beta \geq -\frac{\kappa^2}{2\sigma^2} \quad \text{and} \tag{18}$$

$$\alpha \geq -\frac{\kappa + a}{\sigma^2} \tag{19}$$

*with* $a := \sqrt{\kappa^2 + 2\beta\sigma^2}$. *More precisely, we get*

$$\varphi(t,T,z) = \exp\big( -A(t,T) - B(t,T)\cdot z \big), \tag{20}$$

*where for fixed* $T > 0$ *the functions* $A(\cdot,T)$ *and* $B(\cdot,T)$ *are real-valued* $C^1$*-functions on* $[0,T]$, *which satisfy the ODEs*

$$-\beta - B_t(t,T) + B(t,T)\kappa + 0.5B^2(t,T)\sigma^2 = 0, \tag{21}$$

$$-A_t(t,T) - \kappa\theta B(t,T) = 0 \tag{22}$$

*with* $A(T,T) = 0$ *and* $B(T,T) = \alpha$. *For* $\beta > -\frac{\kappa^2}{2\sigma^2}$ *and* $\alpha > -\frac{\kappa+a}{\sigma^2}$ *the functions* $A$ *and* $B$ *are given by*

$$A(t,T) = -\tfrac{\kappa\theta(\kappa-a)}{\sigma^2}(T-t) + \tfrac{2\kappa\theta}{\sigma^2}\ln\Big(\tfrac{1-ke^{-a(T-t)}}{1-k}\Big), \tag{23}$$

$$B(t,T) = -\frac{-k(\kappa+a)e^{-a(T-t)} + \kappa - a}{\sigma^2(-ke^{-a(T-t)} + 1)}, \tag{24}$$

*with* $k := \frac{\alpha\sigma^2+\kappa-a}{\alpha\sigma^2+\kappa+a}$. *For* $\beta \geq -\frac{\kappa^2}{2\sigma^2}$ *and* $\alpha = -\frac{\kappa+a}{\sigma^2}$ *we obtain*

$$A(t,T) = -\kappa\theta\tfrac{\kappa+a}{\sigma^2}(T-t), \quad B(t,T) = -\tfrac{\kappa+a}{\sigma^2}. \tag{25}$$

**Proof**. See Appendix.

The expected value of $Z$ as well as the function $f$ can be rearranged such that both have a representation corresponding to (17). Therefore, applying the above proposition, we obtain the following result:[6]

**Proposition 5.2 (Sufficient Condition for Well-defined Candidates)** *In Heston's model (14) the process* $Z$ *is a density by all means. Besides, the function* $f$ *is real-valued and has the representation*

$$f(t,z) = \exp\Big( \tfrac{\gamma}{c} r(T-t) - A_f(t,T) - B_f(t,T)z \Big) \tag{26}$$

*if*

$$\frac{\gamma}{1-\gamma}\bar{\lambda}\Big(\frac{\kappa\rho}{\sigma} + \frac{\bar{\lambda}}{2}\Big) < \frac{\kappa^2}{2\sigma^2}. \tag{27}$$

[6] Note that e.g. Novikov's condition would not enable us to conclude that $Z$ is a density for all parametrizations of the model.

*Here, $A_f$ is a real-valued $C^1$-function and $B_f(t,T) = 2\tilde{\beta}\frac{e^{\tilde{a}(T-t)}-1}{e^{\tilde{a}(T-t)}(\tilde{\kappa}+\tilde{a})-\tilde{\kappa}+\tilde{a}}$, where $\tilde{\beta} = -0.5\frac{1}{c}\frac{\gamma}{1-\gamma}\bar{\lambda}^2$, $\tilde{a} = \sqrt{\tilde{\kappa}^2 + 2\tilde{\beta}\sigma^2}$, $c = \frac{1-\gamma}{1-\gamma+\rho^2\gamma}$, and $\tilde{\kappa} = \kappa - \frac{\gamma}{1-\gamma}\rho\bar{\lambda}\sigma$ with*

$$\tilde{\kappa} > 0. \tag{28}$$

**Proof.** See Appendix.

By Proposition 5.2, the candidates for the value function and for the optimal portfolio strategy presented in Section 3 are well-defined under assumption (27). The candidate for the optimal portfolio strategy reads as

$$\begin{aligned} \pi^*(t) &= \frac{1}{1-\gamma}\bar{\lambda} - \frac{1}{1-\gamma}\, c\rho\sigma\, B_f(t,T) \\ &= \frac{1}{1-\gamma}\bar{\lambda} + \frac{\gamma}{(1-\gamma)^2}\,\rho\sigma\bar{\lambda}^2 \frac{e^{\tilde{a}(T-t)}-1}{e^{\tilde{a}(T-t)}(\tilde{\kappa}+\tilde{a})-\tilde{\kappa}+\tilde{a}} \end{aligned} \tag{29}$$

Due to (28) and $\tilde{a} > 0$, the last fraction in (29) is always positive and bounded. Therefore, $\pi^*$ is a deterministic and continuous process, which is consequently bounded on $[0,T]$. Finally, let us remark that for $\gamma < 0$ the function $f$ is always well-defined because the argument of the exponential function in (10) is then negative.

## 5.2 Optimality in Heston's Model

In the previous section we have just seen that, given condition (27) holds, the candidates $G$ and $\pi^*$ are well-defined. Now we will show that this condition does not only provide well-defined candidates, but is also sufficient to verify that the candidates are indeed the value function and the optimal portfolio strategy. Let us remark that the wealth equation reads as

$$dX(t) = X(t)\Big[\Big(r + \bar{\lambda}z(t)\pi(t)\Big)dt + \pi(t)\sqrt{z(t)}\,d\hat{W}_2(t)\Big]. \tag{30}$$

Then we get the following result which is the key result of this paper.

**Theorem 5.1 (Optimality in Heston's Model)** *Assume condition (27) to hold. Then the candidate (29) is the optimal portfolio process in Heston's model (14).*

**Proof.** See Appendix.

Hence, we have found a completely explicit solution of a portfolio problem with an unbounded market price of risk in the Heston setting. Note that due to the special form of the model we obtain the required smoothness for applying Proposition 4.1 without using viscosity theory in contrast to Zariphopoulou (2001) in her setting.

We also want to stress that, by results of Korn/Kraft (2002), the Heston model (14) is (at least partly) not well-behaving if (27) is not satisfied. More precisely, they construct examples where there exists a convex set of *bounded* portfolio strategies leading to *infinite* utility. An obvious hint why such a strange property occurs is the form of $\tilde{a} = \sqrt{\tilde{\kappa}^2 + 2\tilde{\beta}\sigma^2}$ in the representation of the value function: A violation of (27) is equivalent with $\tilde{a}$ being a complex number.

*Remarks:* (i) The property that the portfolio problem does not lead to meaningful results for all parametrizations of the model critically hinges upon the assumption of an unbounded market price of risk. To see this, let us consider a Heston model with a bounded market price of risk. For simplicity, in (14) we choose an excess return of $\lambda(t) = \bar{\lambda} \cdot \sqrt{z(t)}$ (instead of $\lambda(t) = \bar{\lambda} \cdot z(t)$). Then we even obtain a constant market price of risk $\lambda(t)/\nu(t) = \bar{\lambda}$ and, by (10), the function $f$ is always well-defined and does not depend on $z$. Hence, we have $\pi^*(t) = \bar{\lambda}/\sqrt{z(t)}$ and, by applying our results of Section 4, one can easily show that this is the optimal strategy for *every* parametrization of the model. Note that in contrast to our above results the optimal portfolio process $\pi^*$ is now unbounded.

(ii) Assuming $\bar{\lambda} > 0$ condition (27) is equivalent to $\bar{\lambda} < \frac{\kappa}{\sigma}\left(\sqrt{\rho^2 + \frac{1-\gamma}{\gamma}} - \rho\right)$. Hence, (27) gives an upper bound for the slope $\bar{\lambda}$ of the excess return function $\bar{\lambda}_f(t,z) = \bar{\lambda}\cdot z$.

Finally, we apply Theorem 5.1 to prove optimality in a generalized version of Heston's model which for instance is discussed by Liu (2001b). Let us assume that the stock price can be written as

$$dS(t) = S(t)\Big[\Big(r + \bar{\lambda} \cdot [v(t)]^{\frac{1+d}{2}}\Big)dt + \sqrt{v(t)}\, d\hat{W}_2(t)\Big] \tag{31}$$

with $v(t) = [z(t)]^{\frac{1}{d}}$, where $z$ is defined as above and $d \in I\!R \setminus \{0\}$. The Heston model (14) corresponds to the special case $d = 1$ and without loss of generality one can restrict to this case. This can be seen from the wealth equation

$$dX(t) = X(t)\Big[\Big(r + \pi(t) \cdot \bar{\lambda} \cdot [z(t)]^{\frac{1}{2d}+\frac{1}{2}}\Big)dt + \pi(t) \cdot [z(t)]^{\frac{1}{2d}}\, d\hat{W}_2(t)\Big].$$

Applying an idea of Zariphopoulou (2001), the definition $\pi(t) = \hat{\pi}(t)[z(t)]^{-\frac{1}{2d}+\frac{1}{2}}$ leads to

$$dX(t) = X(t)\Big[\Big(r + \hat{\pi}(t) \cdot \bar{\lambda} \cdot z(t)\Big)dt + \hat{\pi}(t) \cdot z(t)^{\frac{1}{2}}\, d\hat{W}_2(t)\Big].$$

Consequently, one can solve the problem using the results of Heston's model (14) and then multiply the optimal proportion by the correction factor $[z(t)]^{-\frac{1}{2d}+\frac{1}{2}}$. We summarize this result in the following corollary:

**Corollary 5.1 (Optimality in the Generalized Heston Model)** *Consider the generalized Heston model (31) and assume (27) to hold. Then the portfolio process* $\pi_c(t) = \pi^*(t) \cdot [z(t)]^{-\frac{1}{2d}+\frac{1}{2}}$ *is the optimal one. Here,* $\pi^*$ *is given by (29).*

# 6 Appendix

**Proof of Proposition 4.1**. To shorten notations, let $X^* = X^{\pi^*}$. Besides, let $\pi$ be an admissible portfolio strategy. Note that the idea for the proof of (12) goes back to Duffie (1996, p. 200).

*Proof of (12).* Since, by assumption (27), $G$ is in $C^{1,2}$, we can apply Ito's formula to obtain

$$\begin{aligned} G(T, X^\pi(T), z(T)) &= G(0, x_0, z_0) \\ &+ \int_0^T A^{\pi(s)} G(s, X^\pi(s), z(s))\, ds + \int_0^T G_z(s, X^\pi(s), z(s))\delta(s)\, dW_1(s) \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T G_x(s,X^\pi(s),z(s))X^\pi(s)\pi(s)\nu(s)\,d\hat{W}_2(s)\\
&\le G(0,x_0,z_0)+\int_0^T G_z(s,X^\pi(s),z(s))\delta(s)\,dW_1(s)\\
&\quad+\int_0^T G_x(s,X^\pi(s),z(s))X^\pi(s)\pi(s)\nu(s)\,d\hat{W}_2(s)\\
&=: Y(T),
\end{aligned}
$$

because, by (9), we have $A^{\pi(s)}G(s,X(s),z(s)) \le 0$ for all $s \in [0,T]$. As $G \ge 0$, the local martingale $Y$ is a supermartingale. Note that $G(T,x,z) = \frac{1}{\gamma}x^\gamma$. Therefore, by taking expectations we obtain (12).

*Proof of (13).* Let

$$
\mathcal{O}_p := [0,\infty)^2 \cap \{w \in \mathbb{R}^2 : |w| < p,\ \mathrm{dist}(w,\partial\mathcal{O}) > p^{-1}\}, \quad p \in \mathbb{N}.
$$

and $Q_p := [0, T-p^{-1}) \times \mathcal{O}_p$, where the sets $Q_p$ are not empty for $p \in \mathbb{N}$ with $p > T^{-1} =: \tilde{p}$. Without loss of generality we therefore assume $p > \tilde{p}$. Besides, let $\theta_p := \min\{T, \tau_p\}$ be a stopping time, where $\tau_p$ denotes the first exit time of $(s, X(s), z(s))$ from $Q_p$. Note that for $p \to \infty$ we get $\tau_p \to \infty$ a.s. and, consequently, $\theta_p \to T$ a.s. As $A^{\pi^*(s)}G(s,X^*(s),z(s)) = 0$ for all $s \in [0,T]$, by Ito's formula, we get

$$
\begin{aligned}
G(\theta_p, X^*(\theta_p), z(\theta_p)) \quad &= \quad G(0,x_0,z_0) + \int_0^{\theta_p} G_z(s,X^\pi(s),z(s))\delta(s)\,dW_1(s)\\
&\quad + \int_0^{\theta_p} G_x(s,X^*(s),z(s))X^*(s)\pi^*(s)\nu(s)\,d\hat{W}_2(s). \qquad (32)
\end{aligned}
$$

As $G_x$ and $G_z$ are continuous, (11) is assumed to hold, and $\pi^* \in \mathcal{A}_2$, the stopped Ito integrals are martingales. Hence, we obtain

$$
\mathrm{E}\Big(G(\theta_p, X^*(\theta_p), z(\theta_p))\Big) = G(0,x_0,z_0).
$$

Since we assume that $\pi^*$ has property U, the family $\Big\{G(\theta_p, X^*(\theta_p), z(\theta_p))\Big\}_p$ is uniformly integrable. Thus, we end up with

$$
G(0,x_0,z_0) = \lim_{p\to\infty} \mathrm{E}\Big(G(\theta_p, X^*(\theta_p), z(\theta_p))\Big) = \mathrm{E}\Big(G(T, X^*(T), z(T))\Big) = \mathrm{E}\Big(\tfrac{1}{\gamma}X^*(T)^\gamma\Big)
$$

and this proves (13). □

**Proof of Proposition 5.1**. We present the relevant parts of a proof given by Kraft (2002). Starting with the ansatz[7]

$$
\varphi(t,T,z) = \exp\Big(-A(t,T) - B(t,T)\cdot z\Big),
$$

where $A(\cdot,T)$ and $B(\cdot,T)$ are continuous differentiable functions of time $t$, one can show that

$$
e^{F(t)} := e^{-\beta\int_0^t z(s)\,ds - A(t,T) - B(t,T)z(t)}
$$

[7]I thank Chris Rogers for pointing out the idea of the proof in the special case when $\alpha$ is zero.

is a $P$-martingale. Applying Ito's formula to the process $F$, we obtain

$$
\begin{aligned}
de^{F(t)} &= e^{F(t)}\Big[\Big\{-\beta - B_t(t,T) + B(t,T)\kappa + 0.5B^2(t,T)\sigma^2\Big\}z(t) \\
&\quad + \Big\{-A_t(t,T) - B(t,T)\kappa\theta\Big\}\Big]dt - e^{F(t)}B(t,T)\sigma\sqrt{z(t)}\,dW_1(t).
\end{aligned}
$$

Hence, $e^F$ is only a martingale if the drift of the above SDE vanishes. This leads to the differential equations (21) and (22) for $A$ and $B$ with $A(T,T) = 0$ and $B(T,T) = \alpha$. We make the ansatz $B(t,T) = -\frac{2}{\sigma^2}\frac{\phi_t(t,T)}{\phi(t,T)}$. Computing the derivative and inserting in (21) leads to

$$
\phi_{tt} - \kappa\phi_t + \tfrac{\beta}{c}\phi = 0. \tag{33}
$$

Solving the corresponding polynomial equation $y^2 - \kappa y + \frac{\beta}{c} = 0$, we obtain

$$
y_{1/2} = 0.5\kappa \pm 0.5\underbrace{\sqrt{\kappa^2 + 2\beta\sigma^2}}_{=:a}.
$$

These solutions are only real numbers if $a \geq 0$ leading to condition (18).

*1st case:* $a > 0$. Then every solution of (33) can be written as

$$
\phi(t) = w_1 e^{0.5(\kappa+a)t} + w_2 e^{0.5(\kappa-a)t}
$$

with $w_1$, $w_2 \in I\!R$. Now $w_1$ and $w_2$ have to be determined according to the terminal condition $B(T,T) = \alpha$. Due to our above ansatz for $B$ we get

$$
B(t,T) = -\frac{w_1(\kappa+a)e^{0.5(\kappa+a)t} + w_2(\kappa-a)e^{0.5(\kappa-a)t}}{\sigma^2(w_1 e^{0.5(\kappa+a)t} + w_2 e^{0.5(\kappa-a)t})}. \tag{34}
$$

Hence, $B(T,T) = a$ leads to

$$
w_1\Big(\alpha + \tfrac{\kappa+a}{\sigma^2}\Big)e^{0.5aT} = -w_2\Big(\alpha + \tfrac{\kappa-a}{\sigma^2}\Big)e^{-0.5aT} \tag{35}
$$

for $w_1$ or $w_2$. If $\alpha = -(\kappa+a)/\sigma^2$, we have $w_2 = 0$ and, consequently, we end up with (25). Assuming $\alpha \neq -(\kappa+a)/\sigma^2$, we obtain

$$
w_1 = -w_2 \underbrace{\frac{\alpha + \frac{\kappa-a}{\sigma^2}}{\alpha + \frac{\kappa+a}{\sigma^2}}}_{=:k} e^{-aT}.
$$

Plugging this result into (34), we get (24). If $k \geq 1$ the denominator of $B$ has a null and, consequently, $B$ is not well-defined. We therefore need conditions which exclude $k \geq 1$. Since $a > 0$, we have $\alpha + \frac{\kappa-a}{\sigma^2} < \alpha + \frac{\kappa+a}{\sigma^2}$. If $\alpha + \frac{\kappa+a}{\sigma^2} > 0$ we get $k < 1$, but if $\alpha + \frac{\kappa+a}{\sigma^2} < 0$ we obtain $k > 1$. Hence, (19) leads to a well-defined function $B$. Integrating (22) we get (23), which is well-defined due to (19).

*2nd case:* $a = 0$. Then all solutions of (33) can be written as

$$
\phi(t) = w_1 e^{0.5\kappa t} + w_2 t e^{0.5\kappa t}
$$

with $w_1$, $w_2 \in I\!R$. If $\alpha = -\kappa/\sigma^2$, we obtain the same result (25) as in the 1st case with $\alpha = -(\kappa+a)/\sigma^2$. Otherwise one can again show that $A$ and $B$ are well-defined real-valued $C^1$-functions. □

**Proof of Proposition 5.2.** The process

$$Z(t) = \exp\Big( -0.5(\tfrac{\gamma}{1-\gamma})^2\rho^2\bar{\lambda}^2 \int_0^t z(s)\,ds + \tfrac{\gamma}{1-\gamma}\rho\bar{\lambda}\int_0^t \sqrt{z(s)}\,dW_1(s)\Big)$$

is at least a supermartingale because it is a positive local martingale. Thus it is sufficient for $Z$ to be a density if $\mathrm{E}(Z(T)) = 1$. Due to the dynamics of the squared volatility (15) we obtain

$$\int_0^t \sigma\sqrt{z(s)}\,dW_1(s) = z(t) - z(0) - \int_0^t \kappa(\theta - r(s))\,ds. \tag{36}$$

Therefore, we get $Z(t) = R(t)\cdot\exp\big(-\frac{\gamma}{1-\gamma}\rho\frac{\bar{\lambda}}{\sigma}(z(0)+\kappa\theta t)\big)$ with

$$R(t) := \exp\Big(\underbrace{\tfrac{\gamma}{1-\gamma}\rho\tfrac{\bar{\lambda}}{\sigma}}_{=:-\alpha_1} z(t) + \int_0^t \underbrace{\Big[\tfrac{\gamma}{1-\gamma}\rho\tfrac{\bar{\lambda}}{\sigma}\kappa - 0.5\Big(\tfrac{\gamma}{1-\gamma}\rho\bar{\lambda}\Big)^2\Big]}_{=:-\beta_1} z(s)\,ds\Big).$$

By Proposition 5.1, the process $R$ has a representation (20) if

$$\beta_1 \geq -\tfrac{\kappa^2}{2\sigma^2}, \tag{37}$$

$$\alpha_1 \geq -\tfrac{\kappa+a_1}{\sigma^2} \tag{38}$$

with $a_1 = \sqrt{\kappa^2 + 2\beta_1\sigma^2}$. As $\kappa^2 + 2\beta_1\sigma^2 = (\kappa - \frac{\gamma}{1-\gamma}\rho\bar{\lambda}\sigma)^2 \geq 0$, the inequality (37) is always satisfied. Further, we get $a_1 = |\kappa - \frac{\gamma}{1-\gamma}\rho\bar{\lambda}\sigma|$. Since the second inequality (38) can be rewritten as $a_1 \geq \frac{\gamma}{1-\gamma}\rho\bar{\lambda}\sigma - \kappa$, it is always met, too. Thus we can apply Proposition 5.1 to $R$. For all parametrizations of the model one can then show that $\mathrm{E}(R(T)) = \exp\big(\frac{\gamma}{1-\gamma}\rho\frac{\bar{\lambda}}{\sigma}(z(0)+\kappa\theta t)\big)$ and this implies $\mathrm{E}(Z(T)) = 1$. Hence, $Z$ is a density by all means. Note that due to the specific structure of $R$ the condition (38) holds as equality if (37) holds as equality. Therefore, we only have to consider the representations given by (23) and (24) as well as by (25).

Now we have to check under which conditions $f$ is a real-valued function having the representation (26). We want to stress that we are now working under the measure $\tilde{P}$ and thus have to use the representation (16) of $z$. The function $f$ can be rewritten as

$$f(t,z) = e^{\frac{\gamma}{c}r(T-t)}\tilde{\mathrm{E}}^{t,z}\Big(e^{-\tilde{\beta}\int_t^T z(s)ds}\Big)$$

with $\tilde{\beta} = -0.5\frac{1}{c}\frac{\gamma}{1-\gamma}\bar{\lambda}^2$ and $c = \frac{1-\gamma}{1-\gamma+\rho^2\gamma}$. Applying Proposition 5.1, $f$ is real-valued and possesses the representation (26) if[8]

$$\tilde{\beta} > -\frac{\tilde{\kappa}^2}{2\sigma^2}, \tag{39}$$

$$0 > -\frac{\tilde{\kappa}+\tilde{a}}{\sigma^2} \tag{40}$$

with $\tilde{a} = \sqrt{\tilde{\kappa}^2 + 2\tilde{\beta}\sigma^2}$. Since $\tilde{\kappa} = \kappa - \frac{\gamma}{1-\gamma}\rho\bar{\lambda}\sigma$, the inequality (39) can be rewritten as

$$\kappa - 2\tfrac{\gamma}{1-\gamma}\rho\bar{\lambda}\sigma > \tfrac{\gamma}{1-\gamma}\tfrac{\bar{\lambda}^2\sigma^2}{\kappa} \tag{41}$$

and this is equivalent to (27). Further, from (41) it follows (28). As $\tilde{a} > 0$, condition (40) is satisfied. The representation of $B_f$ follows from (24). □

[8] Actually, $f$ also has these properties if one or both conditions hold as equality. We do not consider these cases here because we will exclude them later on.

**Proof of Theorem 5.1**. To shorten notations, let $X^* = X^{\pi^*}$, $\tilde{A} = A_f$, and $\tilde{B} = B_f$. Since $\pi^*$ is bounded and (11) holds, it remains to show that $\pi^*$ has property U. To verify this, we show in the sequel that there exists some $q > 1$ such that we have

$$\sup_p \mathrm{E}\Big(|G(\theta_p, X^*(\theta_p), z(\theta_p))|^q\Big) < \infty. \tag{42}$$

By the results of Section 3 and Proposition 5.2, we obtain

$$G(t, X^*(t), z(t)) = \tfrac{1}{\gamma} X^*(t)^\gamma \cdot \Big(f(t, z(t))\Big)^c$$

with $c = \frac{1-\gamma}{1-\gamma+\rho^2\gamma}$ and $f(t,z) = \exp\big(\frac{\gamma}{c} r(T-t) - \tilde{A}(t,T) - \tilde{B}(t,T)z\big)$. Here $\tilde{A}(\cdot,T)$ and $\tilde{B}(\cdot,T)$ are real-valued $C^1$-functions which are bounded on $[0,T]$. Let $q$ be some real number with $q > 1$. Defining $\varepsilon := q - 1 > 0$ we get

$$\begin{aligned}
&|G(t, X^*(t), z(t))|^q \\
&\quad = det(t) \cdot \exp\Bigg[q\gamma \int_0^t \Big(\pi^*(s)\bar{\lambda} - 0.5\pi^*(s)^2\Big) z(s)\, ds + q\gamma \int_0^t \pi^*(s)\sqrt{z(s)}\rho\, dW_1(s) \\
&\qquad + 0.5 q^2\gamma^2 \int_0^t \pi^*(s)^2 z(s)(1-\rho^2)\, ds - qc\tilde{B}(t,T)z(t)\Bigg] \cdot D(t) \\
&\quad = det(t) \cdot \exp\Bigg[\int_0^t q\Big\{0.5\tfrac{\gamma\bar{\lambda}^2 c}{1-\gamma} - 0.5\tfrac{\gamma(1-\gamma)}{c}\Big(\pi^*(s) - \bar{\lambda}\tfrac{c}{1-\gamma}\Big)^2 \\
&\qquad\qquad + 0.5\varepsilon\gamma^2\pi^*(s)^2(1-\rho^2)\Big\} z(s)\, ds + q\gamma\rho \int_0^t \pi^*(s)\sqrt{z(s)}\, dW_1(s) \\
&\qquad\qquad - qc\tilde{B}(t,T)z(t)\Bigg] \cdot D(t)
\end{aligned} \tag{43}$$

with

$$D(t) = \exp\left(-0.5q^2\gamma^2(1-\rho^2)\int_0^t \pi^*(s)^2 z(s)\, ds + q\gamma\sqrt{1-\rho^2}\int_0^t \pi^*(s)\sqrt{z(s)}\, dW_2(s)\right)$$

and $det(t)$ is a positive deterministic term which is bounded on $[0,T]$. By (29), by the SDE of $z$, and by Ito's formula, it follows

$$\begin{aligned}
&\int_0^t \pi^*(s)\sqrt{z(s)}\, dW_1(s) \\
&\qquad = \tfrac{1}{1-\gamma}\tfrac{\bar{\lambda}}{\sigma}\left[z(t) - z(0) - \int_0^t \kappa(\theta - z(s))\, ds\right] \\
&\qquad\quad - \tfrac{c\rho}{1-\gamma}\left[\tilde{B}(t,T)z(t) - \int_0^t \tilde{B}(s,T)\kappa(\theta - z(s))\, ds - \int_0^t \tilde{B}_t(s,T)z(s)\, ds\right],
\end{aligned} \tag{44}$$

where $\tilde{B}_t$ denotes the partial derivative with respect to $t$. Besides, we have

$$\Big(\pi^*(s) - \bar{\lambda}\tfrac{c}{1-\gamma}\Big)^2 = \tfrac{1}{(1-\gamma)^2}\Big((1-c)\bar{\lambda} - c\sigma\rho\tilde{B}(s,T)\Big)^2. \tag{45}$$

Plugging (44) and (45) in (43) leads to

$$\begin{aligned}
&|G(t, X^*(t), z(t))|^q \\
&\quad = det_2(t) \cdot \exp\Bigg[\int_0^t q\Big\{0.5\tfrac{\gamma\bar{\lambda}^2 c}{1-\gamma} - 0.5\tfrac{\gamma}{c(1-\gamma)}\Big((1-c)\bar{\lambda} - c\sigma\rho\tilde{B}(s,T)\Big)^2 \\
&\qquad + 0.5\varepsilon\gamma^2\pi^*(s)^2(1-\rho^2) + \rho\tfrac{\gamma}{1-\gamma}\tfrac{\bar{\lambda}}{\sigma}\kappa - \tfrac{\gamma}{1-\gamma}\rho^2 c\Big(\kappa\tilde{B}(s,T) - \tilde{B}_t(s,T)\Big)\Big\} z(s)\, ds \\
&\qquad + q\Big\{\rho\tfrac{\gamma}{1-\gamma}\tfrac{\bar{\lambda}}{\sigma} - \tilde{B}(t,T)\Big\} z(t)\Bigg] \cdot D(t),
\end{aligned} \tag{46}$$

where "$det_2(t)$" stands for some positive deterministic term which is bounded on $[0,T]$. To proceed, we need some results on the coefficients of the problem which are stated in the following lemma:

**Lemma 6.1** *The following equality holds:*

$$\begin{aligned} 0.5\tfrac{\gamma\bar{\lambda}^2 c}{1-\gamma} - 0.5\tfrac{\gamma}{c(1-\gamma)}\Big((1-c)\bar{\lambda} - c\sigma\rho\tilde{B}(s,T)\Big)^2 + \rho\tfrac{\gamma}{1-\gamma}\tfrac{\bar{\lambda}}{\sigma}\kappa & \\ -\tfrac{\gamma}{1-\gamma}\rho^2 c\Big(\kappa\tilde{B}(s,T) - \tilde{B}_t(s,T)\Big) &= \tfrac{\gamma}{1-\gamma}\bar{\lambda}\Big(\tfrac{\bar{\lambda}}{2} + \tfrac{\kappa\rho}{\sigma}\Big). \end{aligned} \tag{47}$$

*Besides,*

$$\rho\tfrac{\gamma}{1-\gamma}\tfrac{\bar{\lambda}}{\sigma} - \tilde{B}(t,T) < \tfrac{\kappa}{\sigma^2}, \tag{48}$$

*where it is assumed that (27) is satisfied.*

*Proof of (47).* First note that

$$\begin{aligned} &-0.5\tfrac{\gamma}{c(1-\gamma)}\Big((1-c)\bar{\lambda} - c\sigma\rho\tilde{B}(s,T)\Big)^2 \\ &\quad = -0.5\tfrac{\gamma}{1-\gamma}\tfrac{1}{c}(1-c)^2\bar{\lambda}^2 + \tfrac{\gamma}{1-\gamma}(1-c)\bar{\lambda}\sigma\rho\tilde{B}(s,T) - 0.5\tfrac{\gamma}{1-\gamma}c\sigma^2\rho^2\tilde{B}^2(s,T). \end{aligned}$$

Consequently,

$$\begin{aligned} &0.5\tfrac{\gamma\bar{\lambda}^2 c}{1-\gamma} - 0.5\tfrac{\gamma}{1-\gamma}\tfrac{1}{c}(1-c)^2\bar{\lambda}^2 + \tfrac{\gamma}{1-\gamma}(1-c)\bar{\lambda}\sigma\rho\tilde{B}(s,T) - 0.5\tfrac{\gamma}{1-\gamma}c\sigma^2\rho^2\tilde{B}^2(s,T) \\ &+\rho\tfrac{\gamma}{1-\gamma}\tfrac{\bar{\lambda}}{\sigma}\kappa - \tfrac{\gamma}{1-\gamma}\rho^2 c\kappa\tilde{B}(s,T) + \tfrac{\gamma}{1-\gamma}\rho^2 c\tilde{B}_t(s,T) \\ &= \; 0.5\tfrac{\gamma\bar{\lambda}^2 c}{1-\gamma} - 0.5\tfrac{\gamma}{1-\gamma}\tfrac{1}{c}(1-c)^2\bar{\lambda}^2 + \rho\tfrac{\gamma}{1-\gamma}\tfrac{\bar{\lambda}}{\sigma}\kappa \\ &\quad + \tfrac{\gamma}{1-\gamma}\rho^2 c\Big[\underbrace{\tfrac{1-c}{c\rho}\bar{\lambda}\sigma\tilde{B}(s,T) - 0.5\sigma^2\tilde{B}^2(s,T) - \kappa\tilde{B}(s,T) + \tilde{B}_t(s,T) + \tilde{\beta}}_{\overset{(21)}{=}0} - \tilde{\beta}\Big] \\ &= \; \tfrac{\gamma}{1-\gamma}\bar{\lambda}\Big(\tfrac{\bar{\lambda}}{2} + \tfrac{\kappa\rho}{\sigma}\Big). \end{aligned}$$

Note that $\tilde{B}$ meets ODE (21) with $\kappa = \tilde{\kappa}$ and $\beta = \tilde{\beta}$, where $\tilde{\beta} = -0.5\frac{1}{c}\frac{\gamma}{1-\gamma}\bar{\lambda}^2$. Additionally, $\frac{1-c}{c\rho} = \frac{\gamma}{1-\gamma}\rho$, which leads to $\kappa - \frac{1-c}{c\rho}\bar{\lambda}\sigma = \tilde{\kappa}$.

*Proof of (48).* Recall that

$$\tilde{B}(t,T) = 2\tilde{\beta}\frac{e^{\tilde{a}(T-t)} - 1}{e^{\tilde{a}(T-t)}(\tilde{\kappa} + \tilde{a}) - \tilde{\kappa} + \tilde{a}},$$

where $\tilde{\beta} = -0.5\frac{1}{c}\frac{\gamma}{1-\gamma}\bar{\lambda}^2$, $\tilde{a} = \sqrt{\tilde{\kappa}^2 + 2\tilde{\beta}\sigma^2}$, $c = \frac{1-\gamma}{1-\gamma+\rho^2\gamma}$, and $\tilde{\kappa} = \kappa - \frac{\gamma}{1-\gamma}\rho\bar{\lambda}\sigma$. First note that $\tilde{\beta} < 0$ and $-\tilde{B} \geq 0$. For $t \in [0,T)$ we get

$$\tilde{B}(t,T) = 2\tilde{\beta}\frac{1}{\tilde{\kappa} + \tilde{a}h(t)},$$

where $h$ is a deterministic function with $h \geq 1$. By (28) and (39), this leads to the estimate

$$-\tilde{B}(t,T) = 2(-\tilde{\beta})\frac{1}{\tilde{\kappa} + \tilde{a}h(t)} \leq 2(-\tilde{\beta})\frac{1}{\tilde{\kappa}} < \frac{\tilde{\kappa}}{\sigma^2}.$$

Therefore,

$$\rho\tfrac{\gamma}{1-\gamma}\tfrac{\bar{\lambda}}{\sigma} - \tilde{B}(t,T) < \rho\tfrac{\gamma}{1-\gamma}\tfrac{\bar{\lambda}}{\sigma} + \tfrac{\tilde{\kappa}}{\sigma^2} = \tfrac{1}{\sigma^2}\Big(\rho\tfrac{\gamma}{1-\gamma}\bar{\lambda}\sigma + \kappa - \tfrac{\gamma}{1-\gamma}\rho\bar{\lambda}\sigma\Big) = \tfrac{\kappa}{\sigma^2}.$$

Since $\tilde{B}(T,T) = 0$, this relation is also valid for $t = T$ because of (28). This completes the proof of Lemma 6.1.

Now we proceed with the proof of (42). By (47) and (48), we get the following estimate for (46):

$$
\begin{aligned}
&|G(t, X^*(t), z(t))|^q \\
&\overset{(47)}{=} det_2(t) \cdot \exp\Bigg[\int_0^t q\Big\{\tfrac{\gamma}{1-\gamma}\bar{\lambda}\Big(\tfrac{\bar{\lambda}}{2} + \tfrac{\kappa\rho}{\sigma}\Big) + 0.5\varepsilon\gamma^2\pi^*(s)^2(1-\rho^2)\Big\}z(s)\,ds \\
&\qquad\qquad + q\Big\{\rho\tfrac{\gamma}{1-\gamma}\tfrac{\bar{\lambda}}{\sigma} - \tilde{B}(t,T)\Big\}z(t)\Bigg] \cdot D(t) \\
&\overset{(48)}{\le} det_2(t) \cdot \exp\Bigg[\int_0^t q\Big\{\tfrac{\gamma}{1-\gamma}\bar{\lambda}\Big(\tfrac{\bar{\lambda}}{2} + \tfrac{\kappa\rho}{\sigma}\Big) + 0.5\varepsilon\gamma^2\pi^*(s)^2(1-\rho^2)\Big\}z(s)\,ds \\
&\qquad\qquad + q\tfrac{\kappa}{\sigma^2}z(t)\Bigg] \cdot D(t) \\
&= det_2(t) \cdot \exp\Bigg[\int_0^t q\Big\{\tfrac{\gamma}{1-\gamma}\bar{\lambda}\Big(\tfrac{\bar{\lambda}}{2} + \tfrac{\kappa\rho}{\sigma}\Big) + 0.5\varepsilon\gamma^2\pi^*(s)^2(1-\rho^2)\Big\}z(s)\,ds \\
&\qquad\qquad + q\tfrac{\kappa}{\sigma^2}\Big\{z(0) + \kappa\int_0^t(\theta - z(s))\,ds + \sigma\int_0^t\sqrt{z(s)}\,dW_1(s)\Big\}\Bigg] \cdot D(t) \\
&= det_3(t) \cdot \exp\Bigg[\int_0^t q\Big\{\tfrac{\gamma}{1-\gamma}\bar{\lambda}\Big(\tfrac{\bar{\lambda}}{2} + \tfrac{\kappa\rho}{\sigma}\Big) + 0.5\varepsilon\gamma^2\pi^*(s)^2(1-\rho^2)\Big\}z(s)\,ds \\
&\qquad\qquad - q\tfrac{\kappa^2}{\sigma^2}\int_0^t z(s)\,ds + q\tfrac{\kappa}{\sigma}\int_0^t\sqrt{z(s)}\,dW_1(s)\Bigg] \cdot D(t) \\
&= \exp\Bigg[\int_0^t q\Big\{\underbrace{\tfrac{\gamma}{1-\gamma}\bar{\lambda}\Big(\tfrac{\bar{\lambda}}{2} + \tfrac{\kappa\rho}{\sigma}\Big) + 0.5\varepsilon\gamma^2\pi^*(s)^2(1-\rho^2) - 0.5(1-\varepsilon)\tfrac{\kappa^2}{\sigma^2}}_{(*)}\Big\}z(s)\,ds \\
&\qquad\qquad - 0.5(\underbrace{1+\varepsilon}_{=q})q\tfrac{\kappa^2}{\sigma^2}\int_0^t z(s)\,ds + q\tfrac{\kappa}{\sigma}\int_0^t\sqrt{z(s)}\,dW_1(s)\Bigg] \cdot D(t) \cdot det_3(t),
\end{aligned}
$$

where "$det_3(t)$" stands for some positive deterministic term which is bounded on $[0,T]$. Since $\pi^*$ is bounded and, by assumption, (27) holds as strict inequality, we can choose some $\varepsilon > 0$ such that the above term (*) is smaller than zero. To this end, we get

$$
\begin{aligned}
&|G(t, X^*(t), z(t))|^q \\
&\quad \le det_3(t) \cdot \underbrace{\exp\Bigg[-0.5q^2\tfrac{\kappa^2}{\sigma^2}\int_0^t z(s)\,ds + q\tfrac{\kappa}{\sigma}\int_0^t\sqrt{z(s)}\,dW_1(s)\Bigg] \cdot D(t)}_{=:L(t)},
\end{aligned}
$$

where $L$ is (at least) a local martingale which is positive. Therefore, $L$ is a supermartingale. By the optional stopping theorem (OS),[9] we obtain for all stopping times $\theta_p$ with $0 \le \theta_p \le T$

$$
\begin{aligned}
&\mathrm{E}\Big(|G(\theta_p, X^*(\theta_p), z(\theta_p))|^q\Big) \\
&\qquad \le \mathrm{E}\Big(det_3(\theta_p) \cdot L(\theta_p)\Big) \le \sup_{t\in[0,T]} det_3(t) \cdot \mathrm{E}\Big(L(\theta_p)\Big) \overset{OS}{\le} \sup_{t\in[0,T]} det_3(t) < \infty
\end{aligned}
$$

and this proves Theorem 5.1. □

[9] See e.g. Karatzas/Shreve (1991), p. 19.

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