The Futures Market Model and No-Arbitrage Conditions on the Volatility

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Abstract:
Interest rate futures are basic securities and at the same time highly liquid traded objects. Despite this observation, most models of the term structure of interest rate assume forward rates as primary elements. The processes of futures prices are therefore endogenously determined in these models. In addition, in these models hedging strategies are based on forward and/or spot contracts and only to a limited extent on futures contracts.
Inspired by the market model approach of forward rates by Miltersen, Sandmann, and Sondermann (1997), the starting point of this paper is a model of futures prices. Using the prices of futures on interest related assets as the input to the model, new no-arbitrage restricions on the volatility structure are derived. Moreover, these restrictions turn out to prevent an application of a market model based on futures prices.

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Introduction

The aim of this paper is to establish and analyze the no-arbitrage conditions originating from a term structure of interest rate model where the exogenous input consists of futures prices on zero coupon bonds, and the dynamics of these futures prices. The earliest models of the term structure of interest rates appearing in the literature were based on the short-term interest rate as the exogenously given input. The no-arbitrage condition derived claims that the expected excess return of a bond divided by its volatility, should equal the market price of risk function which is found to be independent of the maturity of the bond considered. Assuming the latter function known, interest rate derivatives could then be priced. This modelling was subsequently modified by increasing the set of input to the model. Firstly, the parameters of the stochastic differential equation

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was chosen in such a manner that the model determined prices would be in accordance with the today observed prices. Secondly, the number of state variables were extended, but the no-arbitrage implications on the drift and volatility terms were uninfluenced by these modifications.

A major step forward was made by the so-called Heath, Jarrow, and Morton (1992) modelling approach. Here the dynamics of the family of forward rates and not only the short-term interest rate is the input to the model. This modelling approach highlights the dynamic relationship between different interest rate depending objects, which has to be satisfied in a continuous time dynamic setting without arbitrage, like bonds of different maturities, yields, forward rates, etc. Whereas in earlier models the drift and the volatility terms could be chosen independently of each other, the degree of freedom was now reduced so that only one of these terms could be specified exogenously: an important restriction on the drift parameter in relation to a chosen volatility structure was established.

The strength and elegance of the Heath, Jarrow, and Morton model comes from the exogenous modelling of the instantaneous forward rate processes. However, this is also the most critical aspect of the model: The instantaneous interest rates are theoretical objects defined by taking the limit as the compounding interval approaches zero. These rates do not correspond in any simple way to interest rates observed in real financial markets. One way to exclude negative forward interest rates within the Heath, Jarrow, Morton framework, is to assume a log-normal volatility structure. As pointed out by Hogen and Weintraub (1993), this modelling assumption imposed on instantaneous interest rates implies that rollover returns are infinite. Furthermore, Eurodollar Futures cannot be evaluated within this model specification. It was further argued that this negative result about log-normal term structure modelling takes over to the caplet formula by Black (1976). Therefore, Black's formula was thought to be inconsistent with an arbitrage free model of the term structure of interest rates. As shown by Sandmann and Sondermann (1997), the negative result of log-normal interest rates is a result of modelling continuously compounded rates. Instead, imposing the log-normality assumption on interest rates with finite compounding period, yields a finite expected value of the rollover return. Furthermore, assuming a log-normal structure on nominal interest rates, Miltersen, Sandmann, and Sondermann (1995,1997) justified Black's formula for caps and floors and derived the relationship between what has later been termed the market model approach. ¿From the modelling point of view, the main impact of the market model approach is to shift the objective to nominal forward rates which are closely related to observed market data. Instantaneous forward rates are within this context endogenously determined. However, so far the very liquid futures market has not been analysed to any major extent as the modelling input set.

What happens if we now exogenously specify the much richer family of futures prices? Richer in the sense that a futures has many more payment days than the corresponding forward contract. This question will be addressed in the following. Furthermore, we will analyse whether a consistent market model based on futures prices can be established to overcome the non-adequate behavior of the forward based market model where we cannot in a rational sense price both swaps and caps.

The paper is organized as follows: In Section 1 we recall some known results and present some definitions. Section 2 contains the main model of futures prices related to the term structure of interest rates. In this section new restrictions concerning the volatility structure of the futures price process are presented. In Section 3 we discuss the interrelationship between the Heath, Jarrow, Morton modelling and the futures based approaches. Furthermore, the non-validity of a futures based market model is derived.

1 Spot, Forward and Futures Prices

The construction of an arbitrage free model for financial instruments starts with the definition of the underlying securities. This step is usually not difficult. Nevertheless this modelling step asks for particular care in the case of an interest rate market. There is nothing like the interest rate. Instead, different concepts of interest rates have to be reflected. It has been pointed out by Ingersoll (1987, p. 387) that

> "much of the difficulty with the term structure of interest rates is caused by the cumbersome notation often used. To a less extent, confusion is also caused by an inconsistent use of terminology."

The terminology in this paper includes spot, forward and futures prices as well as the concept of nominal and instantaneous interest rates on spot, forward and futures markets. As a first and careful step we have to fix the notation and to clarify the relationships.
A spot price is the amount of money that we have to pay now (today) to get immediate ownership of a specific good or security. Since we restrict ourselves to the interest rate market, the basic securities are coupon or zero coupon bonds. Setting the face value to one, the owner of a zero coupon bond holds the right to a payment of one unit of account at the maturity of the contract. Denote by $B(t,\tau)$ the spot price at time $t \leq \tau$ of a zero coupon bond with face value one and maturity τ.
In contrast to the spot contract, a forward contract is a binding agreement to deliver a specific good or security at some fixed point in time in the future. Consequently the forward price of a good or security is the amount of money payable at the delivery date. Since, in the case of a forward contract, the closure of the contract and the delivery date are not identical, the forward price differs from the spot price. In particular its dimension is not money today. With respect to the interest rate market we restrict ourselves to forward contracts on zero coupon bonds. Set $t \leq u \leq \tau$ and define by $F(t,u,\tau)$ the forward price of a zero coupon bond with maturity τ and delivery at time u. At time t the forward price $F(t,u,\tau)$ is determined in such a way that the value of the forward contract equals zero. Based on a simple duplication argument, the forward price of a zero coupon bond is related to the spot prices, i.e.

$$F(t,u,\tau) = \frac{B(t,\tau)}{B(t,u)}. \tag{1.1}$$

Closely related to spot prices are spot and forward interest rates. A forward interest rate is an interest rate fixed by two parties for a specific compounding period in the future. Denote by $r_n(t,u,\tau)$ the nominal forward interest rate at time t for the compounding interval $[u, \tau]$ with $t \le u \le \tau$. The relationship between nominal forward rates on the one side and zero coupon bond spot and forward prices on the other side are given by

$$B(t,\tau) \;=:\; \frac{B(t,u)}{(1+(\tau-u)\cdot r_n(t,u,\tau))}, \tag{1.2}$$

$$F(t,u,\tau) \;=:\; \frac{1}{1+(\tau-u)\cdot r_n(t,u,\tau)}. \tag{1.3}$$

The nominal spot rate is obtained for $u = t$. The concept of nominal spot and forward rates is the main modelling instrument of the market model approach. The interest rate concept underlying this class of models is chosen to be close to observable interest rates. In contrast, most other models of the term structure of interest rates are defined on the concept of instantaneous spot and forward rates. Instantaneous interest rates are defined as the limiting concept of nominal interest rates as the length of the compounding period converges to zero. Assume that the spot prices of zero coupon bonds are differentiable with respect to the maturity date. The function of the instantaneous forward rate $f(t,\cdot) : [t,T] \longrightarrow \mathbb{R}$ at time t is defined by:

$$f(t,u) := \lim_{\tau \to u} r_n(t,u,\tau) = -\frac{\partial \ln B(t,u)}{\partial u}. \tag{1.4}$$

As a special case the instantaneous spot rate $r_c(\cdot)$ is defined by:

$$r_c(t) := f(t,t) = \lim_{\tau \to t} r_n(t,t,\tau) = -\left.\frac{\partial \ln B(t,u)}{\partial u}\right|_{u=t}. \tag{1.5}$$

The forward contract as well as the futures contract is an agreement between two parties to exchange a good or security at a specific price in the future. In contrast to the forward contract, the margin system of a futures contract implies a continuous cash flow between the two counter parties. The futures price is fixed in such a way that the value of the futures contract is equal to zero. This implies that the futures price changes over time and the cash flow between the counterparts is determined by the increments of the futures price. Denote by $H(t,u,\tau)$ the futures price at time t if the underlying security is a zero coupon bond with maturity τ, and delivery is at time u. As for the forward price, the dimension of the futures price is not money today. Following Cox, Ingersoll, and Ross (1981), the futures price is equal to the amount of money necessary to implement a self-financing portfolio strategy with a payoff equal to the value of the underlying security times the rollover bank account. Suppose that the underlying security is a zero coupon bond and the marketed-to-market of a futures is continuous. In this case $H(t,u,\tau)$ is equal to the present value of the payoff

$$\exp\left\{\int_t^u r_c(s)ds\right\} B(u,\tau)$$

at time $u \ge t$. This implies that for $u > t$ the forward and futures prices only coincide if the interest rate is deterministic or if $\exp\left\{\int_t^u r_c(s)ds\right\}$ is orthogonal to $B(u,\tau)$. Furthermore, the difference between forward and futures prices is determined by the model of the term structure of interest rates. For $u = t$ we have

$$H(t,t,\tau) = B(t,\tau) = F(t,t,\tau).$$

Finally, with respect to the definition of a futures price, we can now define the implied nominal futures rates. The implied futures rate in nominal terms $r_h(t,u,\tau)$ is defined by:

$$H(t,u,\tau) =: \frac{1}{1+(\tau-u)r_H(t,u,\tau)}. \tag{1.6}$$

This definition of the implied futures rate refers to the case of a Bund Futures contract. In the case of a Eurodollar Futures the implied Eurodollar Futures rate $\tilde{r}_h(t,u,\tau)$ is defined by

$$H(t,u,\tau) =: 1-(\tau-u)\tilde{r}_H(t,u,\tau) \tag{1.7}$$

if $H(t,u,\tau)$ is set equal to the Eurodollar Futures price.
In order to facilitate the notation in the remaining part of the paper, we define the function

$$h(t,u,\tau) := -\frac{\partial \ln H(t,u,\tau)}{\partial \tau} \Longleftrightarrow H(t,u,\tau) = \exp\left\{-\int_u^\tau h(t,u,s)ds\right\}. \tag{1.8}$$

This function has no clear economic interpretation. However, in the limit $u=t$ it turns into the forward rate and for $\tau=u=t$ it consequently turns into the short-term interest rate,

$$\begin{aligned} f(t,\tau) &= h(t,t,\tau) = -\frac{\partial \ln H(t,t,\tau)}{\partial \tau}, \\ r_c(t) &= h(t,t,t) = -\frac{\partial \ln H(t,t,\tau)}{\partial \tau}\bigg|_{\tau=t}. \end{aligned}$$

The function $h(.,.,.)$ has the flavor of an intensity and we will in the future denote it as the intensity of the futures.

2 Stochastic Model of Futures Prices

Any of the mentioned definitions for prices and rates can be used as the starting point of the construction of a model for the term structure of interest rates. Within the Heath, Jarrow and Morton (1992) framework the exogenous assumptions are based on the concept of instantaneous forward rates. Consequently the stochastic evolution of spot and forward prices as well as nominal interest rates are determined endogenously within this model structure. The market model approach by Miltersen, Sandmann and Sondermann (1997) on the other hand is formulated with respect to nominal forward rates. The idea of this section is to use futures prices as the primary and exogenous objects of the modelling structure.
¿From a theoretical point of view one can argue that these approaches are equivalent to each other. Neglecting technical aspects this argument is to some extent valid and will be discussed in this section. Nevertheless two aspects should be stressed at this point.

- First, the information given by futures prices is richer than the one given by forward prices. In addition to the drift restriction determined by the initial forward price curve, further no-arbitrage restrictions should be expected. The intuition is similar to the one of using option prices to calculate the implied state prices, i.e. the martingale measure. In a more practical sense the same idea is applied to calculate implied volatilities. With respect to this, futures contracts are derivatives in the same sense as option contracts and should be used to calculate implied volatilities.

- Second, the futures market is a highly liquid market. Banks, companies and institutional investors are managing large futures positions. Why is a futures based model not used to analyze the risk of these positions? Due to the marketed-to-market system the default risk of futures is less crucial than for forward contracts. To reduce the effects of default risk an empirical study of the term structure of interest rates should refer to the futures market data rather than to the forward market data.

These two aspects serve as the main intuitive justification for the following modelling approach.

2.1 Implied Term Structure of Futures Prices

The futures price is equal to the present value of a self-financing financial strategy with a payoff equal to the value of the underlying security at delivery multiplied by the rollover return. Therefore an arbitrage free model of the term structure of interest rates implies that the futures price is equal to the expected value of the underlying security under the martingale measure. Let $(\Omega, \mathbb{F}, P, \mathbb{F}_t)$ be a filtered probability space and let P^* be a probability measure equivalent to P such that discounted spot price processes are martingales under P^*. The martingale property of spot prices implies for futures prices:

$$\begin{aligned} H(t,u,\tau) &= E_{P^*}[B(u,\tau)|\mathbb{F}_t] = E_{P^*}\left[E_{P^*}[B(u,\tau)|\mathbb{F}_s]|\mathbb{F}_t\right] \\ &= E_{P^*}[H(s,u,\tau)|\mathbb{F}_t] \quad \forall t \leq s \leq u \leq \tau, \end{aligned} \tag{2.1}$$

which yields that the futures price is a martingale under P^*. Note that the martingale property of futures prices is not restricted to zero coupon bonds as underlying securities. For the following we apply the usual modelling framework, i.e. the stochastic processes are defined as stochastic integrals. Assume that the filtration of the probability space is generated by a k-dimensional Brownian motion $\{W^*(t)\}_t$ under the martingale measure P^*. The futures price is therefore a solution of the stochastic differential equation

$$dH(t,u,\tau) = H(t,u,\tau)\sigma_H(t,u,\tau) \cdot dW^*(t), \tag{2.2}$$

where $\sigma_H(\cdot,u,\tau) : [t_0,u] \to \mathbb{R}^k$ is the k-dimensional stochastic volatility function of the futures price. We have to impose restrictions on the volatility structure guaranteeing the existence of the solution to the above stochastic differential equation. In particular, we make the following assumption

Assumption 2.1 *For any tuple (u,τ) with $u \leq \tau$ we assume that the volatility process $\{\sigma_H(t,u,\tau)\}_{t \leq u}$ of the futures price $H(t,u,\tau)$ satisfies the following conditions:*

- *$\{\sigma_H(t,u,\tau)\}_{t\leq u}$ is a k-dimensional continuous and adapted stochastic process with $\sigma_H(t,\tau,\tau) = 0 \quad \forall t \leq \tau$.*

- *The processes of the partial derivatives* $\left\{\frac{\partial \sigma_H(t,u,\tau)}{\partial u}\right\}_{t\le u}$ *and* $\left\{\frac{\partial \sigma_H(t,u,\tau)}{\partial \tau}\right\}_{t\le u}$ *are k-dimensional continuous and adapted processes.*
- $E_{P^*}\left[\exp\left\{\frac{1}{2}\int_t^u ||\sigma_H(s,u,\tau)||^2 ds\right\}\Big| I\!\!F_t\right] < \infty \quad \forall t \le s \le u \le \tau.$
- $E_{P^*}\left[\left|\left|\frac{\partial \sigma_H(s,u,\tau)}{\partial u}\right|\right|^2 \Big| I\!\!F_t\right]$ *and* $E_{P^*}\left[\left|\left|\frac{\partial \sigma_H(s,u,\tau)}{\partial \tau}\right|\right|^2 \Big| I\!\!F_t\right]$ *are bounded on* $t \le s \le u \le \tau$.
- *There exists a predictable and bounded process* $\{A(t,u,\tau)\}_t$ *with* $E_{P^*}[A(s,u,\tau)^2|I\!\!F_t] < \infty \forall s \in [t,u]$ *and* $E_{P^*}[\int_t^u A(s,u,\tau)^2 ds|I\!\!F_t] < \infty$ *such that*

$$\begin{aligned}
\left|\left|\frac{\partial \sigma_H(s,u,\tau)}{\partial u} - \frac{\partial \sigma_H(s,u+\delta,\tau)}{\partial u}\right|\right| &\le A(s,u,\tau)\cdot\delta \quad \forall s \le u \le \tau, \quad \forall \delta > 0 \\
\left|\left|\frac{\partial \sigma_H(s,u,\tau)}{\partial \tau} - \frac{\partial \sigma_H(s,u+\delta,\tau)}{\partial \tau}\right|\right| &\le A(s,u,\tau)\cdot\delta \quad \forall s \le u \le \tau, \quad \forall \delta > 0.
\end{aligned}$$

Furthermore, we assume that at any time t the futures price is continuously differentiable with respect to the delivery date and the maturity date of the underlying zero coupon bond, i.e. $\frac{\partial H(t,u,\tau)}{\partial u}$ and $\frac{\partial H(t,u,\tau)}{\partial \tau}$ exist.

2.2 Implied Term Structure of Forward and Spot Prices

Since the volatility structure of the futures prices is exogenously given, we are interested in the implied structure of spot and forward prices and interest rates. For $u = t$ the futures price is equal to the spot price of the underlying. Therefore the spot price is a solution of the stochastic differential equation

$$\begin{aligned}
dB(t,\tau) &= \left.\frac{\partial H(t,u,\tau)}{\partial u}\right|_{u=t} dt + dH(\cdot,t,\tau)_t \qquad (2.3)\\
&= \left.\frac{\partial H(t,u,\tau)}{\partial u}\right|_{u=t} dt + H(t,t,\tau)\sigma_H(t,t,\tau)\cdot dW^*(t) \\
&= B(t,\tau)\cdot\left.\frac{\partial \ln H(t,u,\tau)}{\partial u}\right|_{u=t} dt + B(t,\tau)\sigma_H(t,t,\tau)\cdot dW^*(t).
\end{aligned}$$

Under the equivalent martingale measure the drift of the zero coupon bond equals the spot rate. Therefore no-arbitrage implies that

$$\left.\frac{\partial \ln H(t,u,\tau)}{\partial u}\right|_{u=t} = r_c(t) \quad \forall t \le \tau. \qquad (2.4)$$

Furthermore, the solution of the futures price process given by equation (2.2) is determined by

$$H(t,u,\tau) = H(t_0,u,\tau)\cdot\exp\left\{-\frac{1}{2}\int_{t_0}^t ||\sigma_H(s,u,\tau)||^2 ds + \int_{t_0}^t \sigma_H(s,u,\tau)\cdot dW^*(s)\right\}. \qquad (2.5)$$

The no-arbitrage condition (2.4) implies the following representation for the instantaneous spot rate process:

$$\begin{aligned} r_c(t) &= \left.\frac{\partial \ln H(t,u,\tau)}{\partial u}\right|_{u=t} \\ &= \frac{\partial \ln H(t_0,t,\tau)}{\partial t} \\ &\quad -\frac{1}{2}\int_{t_0}^{t} \frac{\partial ||\sigma_H(s,t,\tau)||^2}{\partial t} ds + \int_{t_0}^{t} \frac{\partial \sigma_H(s,t,\tau)}{\partial t} \cdot dW^*(s). \end{aligned} \tag{2.6}$$

Taking in (2.6) the expectation given the information at time t_0, we find the following relationship between the expected spot rate under the martingale measure and the futures prices:

$$E_{P^*}\left[r_c(t)|\mathbb{F}_{t_0}\right] = \frac{\partial \ln H(t_0,t,\tau)}{\partial t} - E_{P^*}\left[\int_{t_0}^{t} \sigma_H(s,t,\tau) \cdot \frac{\partial \sigma_H(s,t,\tau)}{\partial t} ds \middle| \mathbb{F}_{t_0}\right]. \tag{2.7}$$

This equation already shows that the expected spot rate and the initial futures prices are closely connected. It still implies a dependency on the volatility structure of the model. Intuition at that point indicates that this model dependency is still too strong, i.e. initial futures prices should completely determine the expected spot rate. To see this we have to consider another way of describing the spot rate, i.e.,

$$r_c(t) = h(t,t,t) := -\left.\frac{\partial \ln H(t,t,\tau)}{\partial \tau}\right|_{\tau=t} \quad \forall t. \tag{2.8}$$

To simplify this second representation of the instantaneous spot rate, notice that, as $\sigma_H(s,t,t) = 0$, we have that

$$\frac{1}{2}\int_{t_0}^{t} \left.\frac{\partial ||\sigma_H(s,t,\tau)||^2}{\partial \tau}\right|_{\tau=t} ds = \int_{t_0}^{t} \sigma_H(s,t,t) \cdot \left.\frac{\partial \sigma_H(s,t,\tau)}{\partial \tau}\right|_{\tau=t} ds = 0 \qquad P^* a.s.$$

With this remark a second representation of the instantaneous spot rate process is derived from equation (2.8) by

$$\begin{aligned} r_c(t) &= -\left.\frac{\partial \ln H(t,t,\tau)}{\partial \tau}\right|_{\tau=t} \\ &= -\left.\frac{\partial \ln H(t_0,t,\tau)}{\partial \tau}\right|_{\tau=t} - \int_{t_0}^{t} \left.\frac{\partial \sigma_H(s,t,\tau)}{\partial \tau}\right|_{\tau=t} \cdot dW^*(s). \end{aligned} \tag{2.9}$$

Taking again expectations, (2.9) yields for the expected spot rate a second relationship:

$$E_{P^*}\left[r_c(t)|\mathbb{F}_{t_0}\right] = -\left.\frac{\partial \ln H(t_0,t,\tau)}{\partial \tau}\right|_{\tau=t} = h(t_0,t,t). \tag{2.10}$$

In other words, the expected spot rate given the information at time t_0 is independent of the volatility structure and completely determined by the initial futures prices. Furthermore, combining the expressions (2.7) and (2.10) gives us a first no-arbitrage condition on the volatility of the term structure of futures prices. At this point of the analysis this condition is still weak, since it is based on the expectation under the martingale measure given the information at time t_0. More precisely any specification of the term structure of volatility for the futures prices has to satisfy the following no-arbitrage condition:

$$\begin{aligned} E_{P^*}\left[\int_{t_0}^{t} \sigma_H(s,t,\tau)\cdot\frac{\partial\sigma_H(s,t,\tau)}{\partial t}ds\,\middle|\,\mathbb{F}_{t_0}\right] &= \frac{\partial\ln H(t_0,t,\tau)}{\partial t}+\left.\frac{\partial\ln H(t_0,t,\tau)}{\partial\tau}\right|_{\tau=t} \\ &= \int_t^{\tau}\frac{\partial^2\ln H(t_0,t,s)}{\partial t\partial s}ds \end{aligned} \tag{2.11}$$

In order to strengthen this condition, we derive the stochastic differential equations of the spot rate implied by the two representations (2.4) and (2.8). Starting with equation (2.4) we have:

$$\begin{aligned} dr_c(t) &= \frac{\partial^2\ln H(t_0,t,\tau)}{\partial t^2}dt-\frac{1}{2}\frac{\partial}{\partial t}\left(\int_{t_0}^{t}\left.\frac{\partial||\sigma_H(s,u,\tau)||^2}{\partial u}\right|_{u=t}ds\right)dt \\ &\quad+\left(\int_{t_0}^{t}\frac{\partial^2\sigma_H(s,t,\tau)}{\partial t^2}\cdot dW^*(s)\right)dt+\left.\frac{\partial\sigma_H(t,u,\tau)}{\partial u}\right|_{u=t}\cdot dW^*(t). \end{aligned} \tag{2.12}$$

The same approach using equation (2.8) yields for the stochastic differential equation for the spot rate process

$$\begin{aligned} dr_c(t) &= -\frac{\partial}{\partial t}\left(\left.\frac{\partial\ln H(t_0,t,\tau)}{\partial\tau}\right|_{\tau=t}\right)dt \\ &\quad-\left(\int_{t_0}^{t}\frac{\partial}{\partial t}\left(\left.\frac{\partial\sigma_H(s,t,\tau)}{\partial\tau}\right|_{\tau=t}\right)\cdot dW^*(s)\right)dt-\left.\frac{\partial\sigma_H(t,t,\tau)}{\partial\tau}\right|_{\tau=t}\cdot dW^*(t). \end{aligned} \tag{2.13}$$

Through matching the two fluctuation parts in equation (2.12) and (2.13), we obtain the following no-arbitrage restriction on the volatility of the futures prices:

$$\left.\frac{\partial\sigma_H(t,u,\tau)}{\partial u}\right|_{u=t}=-\left.\frac{\partial\sigma_H(t,t,\tau)}{\partial\tau}\right|_{\tau=t}. \tag{2.14}$$

In other words, the no-arbitrage conditions (2.4) and (2.9) on futures prices implies that the volatility function must satisfy the same condition. Furthermore, matching the two drift terms results in the following restriction

$$\begin{aligned} \frac{\partial}{\partial t}\left(\int_t^{\tau}\frac{\partial^2\ln H(t_0,t,s)}{\partial t\partial s}ds\right) &= \frac{\partial}{\partial t}\left(\int_{t_0}^{t}\sigma_H(s,t,\tau)\cdot\frac{\partial\sigma_H(s,t,\tau)}{\partial t}ds\right) \\ &\quad-\int_{t_0}^{t}\frac{\partial}{\partial t}\left[\frac{\partial\sigma_H(s,t,\tau)}{\partial t}+\left.\frac{\partial\sigma_H(s,t,\tau)}{\partial\tau}\right|_{\tau=t}\right]\cdot dW^*(s) \\ &= \frac{\partial}{\partial t}\left(\int_{t_0}^{t}\sigma_H(s,t,\tau)\cdot\frac{\partial\sigma_H(s,t,\tau)}{\partial t}ds\right) \\ &\quad-\frac{\partial}{\partial t}\left(\int_{t_0}^{t}\left[\frac{\partial\sigma_H(s,t,\tau)}{\partial t}+\left.\frac{\partial\sigma_H(s,t,\tau)}{\partial\tau}\right|_{\tau=t}\right]\cdot dW^*(s)\right). \end{aligned} \tag{2.15}$$

where the last equality is derived by applying the no-arbitrage condition (2.14). This second no-arbitrage condition on the volatility of the futures prices is more general than the condition (2.11), i.e. taking expectations on both sides implies the former condition. We sum up the results in the following Proposition

Proposition 2.2 *Suppose that the volatility function of the futures price satisfies Assumption 2.1 then:*

i) Under the martingale measure the expected value of the instantaneous spot rate equals the appropriate intensity of the futures at t_0, i.e.

$$E_{P^*}\left[r_c(t)|\mathbb{F}_{t_0}\right] = h(t_0,t,t) = -\left.\frac{\partial \ln H(t_0,t,\tau)}{\partial \tau}\right|_{\tau=t}.$$

ii) No-arbitrage conditions imply that the volatility of the futures price must satisfy a.s.

$$\left.\frac{\partial \sigma_H(t,u,\tau)}{\partial u}\right|_{u=t} = -\left.\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right|_{\tau=t}.$$

and furthermore

$$\begin{aligned}\frac{\partial}{\partial t}\left(\int_t^\tau \frac{\partial^2 \ln H(t_0,t,s)}{\partial t \partial s} ds\right) &= \frac{\partial}{\partial t}\left(\int_{t_0}^t \sigma_H(s,t,\tau)\cdot\frac{\partial \sigma_H(s,t,\tau)}{\partial t} ds\right) \\ &\quad -\frac{\partial}{\partial t}\left(\int_{t_0}^t \left[\frac{\partial \sigma_H(s,t,\tau)}{\partial t} + \left.\frac{\partial \sigma_H(s,t,\tau)}{\partial \tau}\right|_{\tau=t}\right]\cdot dW^*(s)\right).\end{aligned}$$

iii) A weaker condition, derived by taking expectations in (2.15) and applying (2.14), is

$$\int_t^\tau \frac{\partial^2 \ln H(t_0,t,s)}{\partial t \partial s} ds = E_{P^*}\left[\left.\int_{t_0}^t \sigma_H(s,t,\tau)\cdot\frac{\partial \sigma_H(s,t,\tau)}{\partial t} ds\right| \mathbb{F}_{t_0}\right].$$

The statements in Proposition 2.2 concentrate on two aspects. First, the initial futures prices or more precisely the futures intensity completely determine the expected spot rate under the martingale measure. Second, the volatility function cannot be specified in an arbitrary manner. The formal relationship between the initial futures intensity and the volatility is given by Proposition 2.2. Unfortunately, there is no easy interpretation possible. The weaker condition iii) implies that the expected volatility under the martingale measure is determined by the initial futures prices. Under the additional assumption of a deterministic volatility structure, this immediately implies that the volatility structure is completely determined by the initial futures prices. In this specific situation the intuition mentioned in Section 2 is fully reflected. The futures prices and the call option prices for all exercise prices contain information sufficient to generate respectively the volatility structure of the futures prices and the risk-neutral probability measure. Furthermore, from an empirical point of view any sufficiently smooth approximation of futures prices is equivalent to a complete specification of the volatility surface. Without the assumption of a deterministic volatility structure, the mentioned relationship only holds in expectation. In other words, the drift of the volatility is restricted by the initial futures prices. To further understand the no-arbitrage restriction given in Proposition 2.2 we have to consider specific situations. As a natural first guess on the volatility function, we choose an additive separable structure given as

$$\sigma_H(s,t,\tau) = \nu(s,\tau) - \nu(s,t), \tag{2.16}$$

where the function $\nu(.,.)$ is sufficiently differentiable in the second time parameter. With this choice the condition (2.15) reduces to

$$\begin{aligned} \frac{\partial}{\partial t}\left(\int_t^\tau \frac{\partial^2 \ln H(t_0,t,s)}{\partial t \partial s} ds\right) &= -\left(\nu(t,\tau) - \nu(t,t)\right) \cdot \left.\frac{\partial \nu(t,\tau)}{\partial \tau}\right|_{\tau=t} \\ &\quad - \int_{t_0}^t \left((\nu(s,\tau) - \nu(s,t)) \cdot \frac{\partial^2 \nu(s,t)}{\partial t^2} - \left\| \frac{\partial \nu(s,t)}{\partial t} \right\|^2 \right) ds. \end{aligned}$$

As the left hand side is $\mathbb{F}_{t_0}$-measurable, the function $\nu(.,.)$ on the right hand side has to be a deterministic function. Furthermore, assuming additive separability for the futures price volatility, implies that the forwards price and the futures price volatility are the same. This very specific feature is only sufficient to satisfy the no-arbitrage conditions in Proposition 2.2. As we will show in the next section this specific form is not necessary.

2.3 Implications of deterministic volatility functions

If it is assumed that the volatility of the futures price is deterministic, further restrictions can be established on the volatility function. Considering (2.15) we obtain that the volatility of the stochastic process

$$V(t) = -\frac{\partial}{\partial t}\left(\int_{t_0}^t \left[\frac{\partial \sigma_H(s,t,\tau)}{\partial t} + \left.\frac{\partial \sigma_H(s,t,\tau)}{\partial \tau}\right|_{\tau=t}\right] \cdot dW^*(s)\right) \tag{2.17}$$

must be equal to zero. As

$$\begin{aligned} dV(t) &= -\frac{\partial}{\partial u}\left[\frac{\partial \sigma_H(t,u,\tau)}{\partial u} + \left.\frac{\partial \sigma_H(t,u,\tau)}{\partial \tau}\right|_{\tau=u}\right]\Bigg|_{u=t} \cdot dW^*(t) \\ &\quad - \int_{t_0}^t \frac{\partial^2}{\partial t^2}\left[\frac{\partial \sigma_H(s,t,\tau)}{\partial t} + \left.\frac{\partial \sigma_H(s,t,\tau)}{\partial \tau}\right|_{\tau=t}\right] \cdot dW^*(s) dt \end{aligned} \tag{2.18}$$

we find that

$$0 = -\frac{\partial}{\partial u}\left[\frac{\partial \sigma_H(t,u,\tau)}{\partial u} + \left.\frac{\partial \sigma_H(t,u,\tau)}{\partial \tau}\right|_{\tau=u}\right]\Bigg|_{u=t}. \tag{2.19}$$

This is a restriction on the "acceleration" of the volatility function in the point (t,t,t)

$$0 = \left.\frac{\partial^2 \sigma_H(t,u,\tau)}{\partial u^2}\right|_{u=t} + \left.\frac{\partial^2 \sigma_H(t,u,\tau)}{\partial u \partial \tau}\right|_{\tau=t,u=t} + \left.\frac{\partial^2 \sigma_H(t,t,\tau)}{\partial \tau^2}\right|_{\tau=t} \tag{2.20}$$

and furthermore, it tells us that

$$\left.\frac{\partial^2 \sigma_H(t,u,\tau)}{\partial u^2}\right|_{u=t}$$

is independent of τ.
An immediate extension of the additive separable volatility function discussed in the last section is

$$\sigma_H(t,u,\tau) = \nu(t,\tau) - \nu(t,u) + \sigma \cdot (\tau - u)^\gamma \cdot (u-t)^\alpha,$$

where σ, α and γ are some non-negative constants with $\alpha \neq 1, 2$. Differentiation of the volatility function shows that the no-arbitrage conditions in Proposition 2.2 are satisfied. In addition this function implies that the volatility of the process $\{V(t)\}_t$ defined by (2.17) is equal to zero a.s. As an example we consider an application of this structure leading to the Vasicek model.

Example 2.3 *Suppose that the intensity of the futures at time t_0 is up to some constant determined by the following function:*

$$h(t_0,t,\tau) = h(t_0,t_0,\tau) + \sum_{i=1}^{k} \eta_i e^{-\beta_i(\tau-t_0)} \cdot [\cosh(\beta_i(t-t_0)) - 1],$$

where $h(t_0,t_0,\tau) = f(t_0,\tau)$ is equal to the instantaneous forward rate. $\eta = (\eta_1, ...\eta_k)$ and $\beta := (\beta_1, ..., \beta_k)$ are k-dimensional constants. In addition, suppose that the volatility is deterministic. Applying Proposition 2.2 the volatility of the futures price has to satisfy

$$\begin{aligned}
\int_{t_0}^{t} \sigma_H(s,t,\tau) \cdot \frac{\partial \sigma_H(s,t,\tau)}{\partial t} ds &= \int_t^\tau -\frac{\partial h(t_0,t,s)}{\partial t} ds \\
&= \int_t^\tau -\sum_{i=1}^{k} \eta_i \cdot \beta_i e^{-\beta_i(s-t_0)} \cdot \sinh(\beta_i(t-t_0)) ds \\
&= -\sum_{i=1}^{k} \eta_i \cdot \sinh(\beta_i(t-t_0)) \left(e^{-\beta_i(t-t_0)} - e^{-\beta_i(\tau-t_0)}\right) \\
&= -\sum_{i=1}^{k} \frac{\eta_i}{2} \left[e^{-\beta_i t} - e^{-\beta_i \tau}\right] \cdot e^{-\beta_i t} \cdot \left[e^{2\beta_i t} - e^{2\beta_i t_0}\right] \\
&= -\sum_{i=1}^{k} \eta_i \beta_i \left[e^{-\beta_i t} - e^{-\beta_i \tau}\right] \cdot e^{-\beta_i t} \cdot \int_{t_0}^{t} e^{2\beta_i s} ds \\
&= -\sum_{i=1}^{k} \eta_i \beta_i \int_{t_0}^{t} \left[e^{-\beta_i(t-s)} - e^{-\beta_i(\tau-s)}\right] \cdot e^{-\beta_i(t-s)} ds
\end{aligned}$$

This implies that the volatility function should be of the form:

$$\sigma_H(s,t,\tau) = \left(\sqrt{\eta_1}\left(e^{-\beta_1(t-s)} - e^{-\beta_1(\tau-s)}\right), ..., \sqrt{\eta_k}\left(e^{-\beta_k(t-s)} - e^{-\beta_k(\tau-s)}\right)\right)$$

i.e. the above specification of the initial intensity of futures and the assumption of a deterministic volatility structure imply that the term structure of interest rates is determined by a k-factor, non-Markovian Vasicek type term structure model with the speed factor $\beta := (\beta_1, ..., \beta_k)$ and the volatility of the instantaneous spot rate equal to

$$\sigma := \left(\beta_1 \cdot \sqrt{\eta_1}, ..., \beta_k \cdot \sqrt{\eta_k}\right).$$

In other words Proposition 2.2 enables us to estimate the volatility structure from the initial futures prices in the deterministic case. If the volatility is stochastic, the initial futures prices restrict the volatility structure. In the Vasicek case the solution for the spot rate is given by

$$r_c(t) = h(t_0, t, t) - \sum_{i=1}^{k} \int_{t_0}^{t} \beta_i \sqrt{\eta_i} e^{-\beta_i (t-s)} dW_i^*(s).$$

In a similar way, a continuous time version of the Ho and Lee (1986) term structure model can be found. Since the Ho and Lee model can be expressed as the limit of the Vasicek model, if the speed factor approches zero, the corresponding intensity for the Ho and Lee model at time t_0 is up to some constant given by

$$h(t_0, t, \tau) = h(t_0, t_0, \tau).$$

Furthermore, due to the Markovian structure of the model, it is sufficient to consider the 1-factor case. Therefore the futures volatility function is determined by

$$\sigma_H(s, t, \tau) = \sigma(\tau - t)$$

and the continuously compounded spot rate by

$$r_c(t) = h(t_0, t, t) + \sigma \left(W^*(t) - W^*(t_0) \right).$$

3 HJM and the Futures Market model

The intensity of the futures and the forward rates can be expressed as respectively

$$\begin{aligned} f(t,\tau) &= -\frac{\partial \ln H(t,t,\tau)}{\partial \tau} \qquad (3.1) \\ &= -\frac{\partial \ln H(t_0,t,\tau)}{\partial \tau} + \frac{1}{2} \int_{t_0}^{t} \frac{\partial ||\sigma_H(s,t,\tau)||^2}{\partial \tau} ds - \int_{t_0}^{t} \frac{\partial \sigma_H(s,t,\tau)}{\partial \tau} \cdot dW^*(s), \end{aligned}$$

and

$$\begin{aligned} h(t,u,\tau) &= -\frac{\partial \ln H(t,u,\tau)}{\partial \tau} \qquad (3.2) \\ &= -\frac{\partial \ln H(t_0,u,\tau)}{\partial \tau} + \frac{1}{2} \int_{t_0}^{t} \frac{\partial ||\sigma_H(s,u,\tau)||^2}{\partial \tau} ds - \int_{t_0}^{t} \frac{\partial \sigma_H(s,u,\tau)}{\partial \tau} \cdot dW^*(s). \end{aligned}$$

In addition to the structure of the intensity of the futures the specification of the futures model endogenously fixes the dynamics of the forward prices and the nominal forward and futures rates. Applying Itô‘s Lemma, the forward price is given by:

$$\begin{aligned} dF(t,u,\tau) &= d\left(\frac{B(t,\tau)}{B(t,u)}\right) \qquad (3.3) \\ &= F(t,u,\tau)\left(\sigma_H(t,t,\tau) - \sigma_H(t,t,u)\right) \cdot \left(dW^*(t) - \sigma_H(t,t,u)dt\right) \\ &= F(t,u,\tau)\left(\sigma_H(t,t,\tau) - \sigma_H(t,t,u)\right) \cdot dW^u(t), \end{aligned}$$

where $dW^u(t) := dW^*(t) - \sigma_H(t,t,u)dt$ defines a Brownian motion under the u-forward risk adjusted measure Q^u

$$\left.\frac{dQ^u}{dP^*}\right|_t := \frac{\exp\left\{-\int_t^u r_c(s)ds\right\} B(u,u)}{E_{P^*}\left[\exp\left\{-\int_t^u r_c(s)du\right\} B(u,u)\middle| \mathbb{F}_t\right]} = \exp\left\{-\frac{1}{2}\int_t^u ||\sigma_H(s,s,u)||^2 ds + \int_t^u \sigma_H(s,s,u)\cdot dW^*(s)\right\} . \tag{3.4}$$

Since the forward price process $\{F(t,u,\tau)\}_t$ is a martingale under the u-forward risk adjusted measure, the nominal forward rate, $\{r_n(t,u,\tau)\}_t$, is a martingale under the τ-forward risk adjusted measure, i.e.

$$\begin{aligned} dr_n(t,u,\tau) &= d\frac{1}{\tau-u}\left(F(t,u,\tau)^{-1}-1\right) \\ &= -\frac{1}{\tau-u}\frac{1}{F(t,u,\tau)^2}dF(t,u,\tau) + \frac{1}{\tau-u}\frac{1}{F(t,u,\tau)^3}d < F(\cdot,u,\tau) >_t \\ &= -\frac{r_n(t,u,\tau)}{1-F(t,u,\tau)}\left(\sigma_H(t,t,\tau)-\sigma_H(t,t,u)\right)\cdot\left(dW^*(t)-\sigma_H(t,t,\tau)dt\right) \\ &= -\frac{r_n(t,u,\tau)}{1-F(t,u,\tau)}\left(\sigma_H(t,t,\tau)-\sigma_H(t,t,u)\right)\cdot dW^\tau(t). \end{aligned} \tag{3.5}$$

Similarly, we know that the futures price process $(H(t,u,\tau))_t$ is a martingale under P^*. Therefore the process of the implied futures rate defined by

$$r_H(t,u,\tau) := \frac{1}{\tau-u}\left(H(t,u,\tau)^{-1}-1\right)$$

is a solution to the following stochastic differential equation

$$dr_H(t,u,\tau) = -\frac{r_H(t,u,\tau)}{1-H(t,u,\tau)}\left(\sigma_H(t,u,\tau)\cdot dW^*(t) - ||\sigma_H(t,u,\tau)||^2 dt\right). \tag{3.6}$$

The process of the implied futures rate $(r_H(t,\tau,\alpha))_t$ is not a martingale under P^*. Define for $t \le u \le \tau$ by

$$\left.\frac{dQ_H^{u,\tau}}{dP^*}\right|_t = \frac{H(u,u,\tau)}{E_{P^*}[H(u,u,\tau)|\mathbb{F}_t]} = \exp\left\{-\frac{1}{2}\int_t^u ||\sigma_H(s,u,\tau)||^2 du + \int_t^u \sigma_H(s,u,\tau)dW^*(u)\right\} \tag{3.7}$$

a new probability measure. Under this (u,τ)-futures risk adjusted measure, $Q_H^{u,\tau}$, the process $(W_H^{u,\tau}(t))_t$ with

$$dW_H^{u,\tau}(t) := dW^* - \sigma_H(t,u,\tau)\cdot dt \tag{3.8}$$

is a standard Brownian motion and the implied futures rate is a martingale, i.e.

$$dr_H(t,u,\tau) = -\frac{r_H(t,u,\tau)}{1-H(t,u,\tau)}\sigma_H(t,u,\tau)\cdot dW_H^{u,\tau}(t). \tag{3.9}$$

The martingale property of the implied futures rate corresponds to a change of numeraire. In the case of the forward rate adjusted measure a zero coupon bond is chosen to be the numeraire. The futures risk adjusted measure is derived by choosing a futures price as the numeraire. Note that, for $u = t$, the (u, τ)-futures risk adjusted measure and the τ-forward risk adjusted measure coincide. Under the futures risk adjusted measure the solution for the intensity of the futures price in equation (3.2) can be expressed as:

$$\begin{aligned} h(t,u,\tau) &= -\frac{\partial \ln H(t_0,u,\tau)}{\partial \tau} - \int_{t_0}^{t} \frac{\partial \sigma_H(s,u,\tau)}{\partial \tau} \cdot dW_H^{u,\tau}(s), \\ f(t,\tau) &= -\frac{\partial \ln H(t_0,t,\tau)}{\partial \tau} - \int_{t_0}^{t} \frac{\partial \sigma_H(s,t,\tau)}{\partial \tau} \cdot dW_H^{t,\tau}(s), \\ r_c(t) &= \frac{\partial \ln H(t_0,t,\tau)}{\partial t} + \int_{t_0}^{t} \frac{\partial \sigma_H(s,t,\tau)}{\partial t} \cdot dW_H^{t,\tau}(s). \end{aligned} \tag{3.10}$$

This implies that the intensity of the futures price is a martingale under the appropriate futures risk adjusted measure, i.e.

$$dh(t,u,\tau) = \left(-\frac{\partial \sigma_H(t,u,\tau)}{\partial \tau}\right) \cdot dW_H^{u,\tau}(t). \tag{3.11}$$

The instantaneous forward rate can be expressed as a solution to the following stochastic differential equation:

$$\begin{aligned} df(t,\tau) &= dh(t,t,\tau) + \left.\frac{\partial h(t,u,\tau)}{\partial u}\right|_{u=t} dt \qquad (3.12) \\ &= \left(-\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right) \cdot dW_H^{t,\tau}(t) + \left.\frac{\partial h(t,u,\tau)}{\partial u}\right|_{u=t} dt \\ &= \left(-\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right) \cdot dW^{\tau}(t) - \frac{\partial}{\partial \tau}\left(\left.\frac{\partial \ln H(t,u,\tau)}{\partial u}\right|_{u=t}\right) dt \\ &= \left(-\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right) \cdot dW^{\tau}(t) - \frac{\partial}{\partial \tau}\left(r_c(t)\right) dt \\ &= \left(-\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right) \cdot dW^{\tau}(t). \end{aligned}$$

The instantaneous forward rate is a martingale under the appropriate forward risk adjusted measure. Similarly, the spot rate is a solution to

$$\begin{aligned} dr_c(t) &= df(t,t) + \left.\frac{\partial f(t,\tau)}{\partial \tau}\right|_{\tau=t} dt \qquad (3.13) \\ &= \left(-\left.\frac{\partial \sigma_H(t,t,\tau)}{\partial \tau}\right|_{\tau=t}\right) \cdot dW^{*}(t) - \left(\left.\frac{\partial^2 \ln H(t,t,\tau)}{\partial \tau^2}\right|_{\tau=t}\right) dt. \end{aligned}$$

Example 3.1 *Consider again the example of the Vasicek term structure model, i.e. assume that the intensity of the futures prices at time t_0 are determined by:*

$$h(t_0,t,\tau) = h(t_0,t_0,\tau) + \eta e^{-\beta(\tau - t_0)} \cdot [\cosh(\beta(t-t_0)) - 1],$$

where for simplicity we use the 1-factor version. ¿From equation (3.11) the intensity is a solution to the stochastic differential equation

$$dh(t,u,\tau) = -\beta\sqrt{\eta}e^{-\beta(\tau-t)}dW_H^{u,\tau}(t).$$

Similarly, the forward rate can be expressed by

$$df(t,\tau) = -\beta\sqrt{\eta}e^{-\beta(\tau-t)}dW^{\tau}(t).$$

Furthermore, applying equation(3.13) implies the usual representation of the spot rate

$$dr_c(t) = \beta[\theta(t) - r_c(t)]dt + \beta\sqrt{\eta}dW^*(t),$$

with

$$\theta(t) := h(t_0,t,t) + \frac{1}{\beta}\left.\frac{\partial h(t_0,t,\tau)}{\partial \tau}\right|_{\tau=t} + \frac{\beta\eta}{2}\left(1 - e^{-2\beta(t-t_0)}\right).$$

3.1 A Futures Market Model and Log-normality?

Within this section we consider a specific volatility structure which is in accordance with the LIBOR-Market model, i.e.

$$\sigma_H(t,u,\tau) = (1 - H(t,u,\tau)) \cdot \gamma_H(t,u,\tau) \qquad \forall t \le u \tag{3.14}$$

where the functions $\gamma_H(\cdot,u,\tau) : [t_0,u] \to \mathbb{R}^k$ are $\forall u \in [t_0,T]$ and $\forall \tau \in [t_0,T]$ with $u \le \tau$ deterministic and bounded. With this choice the implied futures rate process can be rewritten as the solution of the following stochastic differential equation:

$$\begin{aligned} dr_H(t,u,\tau) &= -r_H(t,u,\tau)\gamma_H(t,u,\tau) \cdot dW_H^{u,\tau}(t) \\ &= -r_H(t,u,\tau)\gamma_H(t,u,\tau) \\ &\quad \cdot \left(dW^*(t) - (1 - H(t,u,\tau))\gamma_H(t,u,\tau) \cdot dt\right). \end{aligned} \tag{3.15}$$

The nominal futures rate processes are log-normal martingales under the appropriate futures risk adjusted measure. Obviously the futures market structure is similar to the modelling assumptions within the LIBOR market model. Nevertheless, defining the volatility of the futures market in accordance with equation (3.14), implies that no forward rate process is a log-normal martingale under the appropriate forward risk adjusted measure. The log-normality for the corresponding nominal forward rate process under the τ-forward risk adjusted measure is fulfilled if a deterministic function $\gamma_F(.,u,\tau)$ exists such that

$$\sigma_H(t,t,\tau) - \sigma_H(t,t,u) = (1 - F(t,u,\tau)) \cdot \gamma_F(t,u,\tau).$$

Instead, the specification (3.14) implies that this function should equal

$$\gamma_F(t,u,\tau) = \frac{(1 - B(t,\tau))\gamma_H(t,t,\tau) - (1 - B(t,u))\gamma_H(t,t,u)}{B(t,\tau) - B(t,u)} \cdot B(t,u).$$

Even if we assume $\gamma_H(.,.,.)$ to be constant, a stochastic specification of the function $\gamma_F(.,.,.)$ is implied. In other words, the Black formula for caplets and floorlets are not satisfied. The reason for this strong property is that, assuming that (3.14) is valid for all τ, is a much stronger assumption than the corresponding one in the LIBOR market model. In particular log-normality within the LIBOR market model can only be imposed on one specific compounding period $\tau - u$. As we will see the same property holds within the futures market model.

We will now analyze whether the above log-normal volatility structure satisfies the conditions in Proposition 2.2. Inserting the expression for $\sigma_H(t,u,\tau)$ into (2.14), leads to the condition

$$- \left.\frac{\partial H(t,u,\tau)}{\partial u}\right|_{u=t} \gamma_H(t,t,\tau) + (1 - H(t,t,\tau)) \left.\frac{\partial \gamma_H(t,u,\tau)}{\partial u}\right|_{u=t} = \left.\frac{\partial H(t,t,\tau)}{\partial \tau}\right|_{\tau=t} \gamma_H(t,t,t)$$

which can be reformulated to

$$r_c(t)(B(t,\tau)\gamma_H(t,t,\tau) - \gamma_H(t,t,t)) = (1 - B(t,\tau)) \left.\frac{\partial \gamma_H(t,u,\tau)}{\partial u}\right|_{u=t}.$$

It is obvious that not even a constant γ_H-function is a valid specification. Furthermore, it seems to be implausible that any deterministic γ_H-function can satisfy this condition. Therefore it is not necessary to consider the second no-arbitrage constrained given in Proposition 2.2 ii). We can safely conclude that a model specification for all compounding periods chosen in accordance with the LIBOR Market model is not an arbitrage free futures market specification.

3.2 Closed form Option Pricing within a restricted Futures Market Model

¿From Section 3 we know that under the (u,τ)-futures risk adjusted measure $Q_H^{u,\tau}$, the process $(W_H^{u,\tau})_t$ with $dW_H^{u,\tau}(t) := dW^*(t) - \sigma_H(t,u,\tau) \cdot dt$ is a standard Brownian motion and the implied futures rate is a martingale. Furthermore, we know that a log-normal volatility structure for all compounding periods $(\tau - u)$ is not arbitrage free. If we fix one specific compounding period $(\tau^* - u^*)$, we can impose the log-normal assumption (3.14) on this specific compounding period, i.e we can assume

$$dr_H(t,u^*,\tau^*) = -r_H(t,u^*,\tau^*)\gamma_H(t,u^*,\tau^*) \cdot dW_H^{u^*,\tau^+}(t). \tag{3.16}$$

where $\gamma_H(\cdot,u^*,\tau^*) : [t_0,u^*] \to \mathbb{R}^k$ is a deterministic function. The implied futures rate $\{r_H(,u^*,\tau^*)\}_t$ is a log-normal martingale under the appropriate futures risk adjusted measure. In this case the no-arbitrage value of futures style options on the implied futures rate or the futures price are easy to calculate.

Consider the futures price of a futures style option on the implied futures rate. The no arbitrage price of the option with face value V, strike L on the implied futures rate

$r_H(t,u^*,\tau^*)$ and maturity $s \geq t$ is equal to:

$$\begin{aligned}
& V(\tau^*-u^*)E_{P^*}\left[[r_H(s,u^*,\tau^*)-L]^+|\mathbb{F}_t\right] \qquad (3.17)\\
& V(\tau^*-u^*)H(t,u^*,\tau^*)E_{Q_H^{u^*,\tau^*}}\left[H(s,u^*,\tau^*)^{-1}[r_H(s,u^*,\tau^*)-L]^+|\mathbb{F}_t\right]\\
& V(\tau^*-u^*)H(t,u^*,\tau^*)E_{Q_H^{u^*,\tau^*}}\left[(1+(\tau^*-u^*)r_H(s,u^*,\tau^*))[r_H(s,u^*,\tau^*)-L]^+|\mathbb{F}_t\right]\\
= & V(\tau^*-u^*)H(t,u^*,\tau^*)\Big[r_H(t,u^*,\tau^*)N(d_1)-LN(d_2)\\
& \qquad +(\tau^*-u^*)r_H(t,u^*,\tau^*)\Big(r_H(t,u^*,\tau^*)e^{g^2}N(d_1+g)-LN(d_1)\Big)\Big]
\end{aligned}$$

with

$$d_{1/2} := \frac{\ln\left(\frac{r_H(t,u^*,\tau^*)}{L}\right) \pm \frac{1}{2}g^2}{g} \quad \text{and} \quad g^2 := \int_t^s ||\gamma_H(\theta,u^*,\tau^*)||^2 d\theta.$$

Formally the pricing formula of a futures style option on the implied futures rate corresponds to a sum of two caplet formulas. The main reason for this is that the implied futures rate is a martingale under the futures risk adjusted measure and not under the martingale measure. Furthermore, the margin system implies that the futures style option is equivalent to a contract situation with payment in advance.

The pricing formula (3.17) is based on the definition of the nominal futures rate. This corresponds not to the definition of the Eurodollar Futures rate. The implied Eurodollar Futures rate is by definition linearly related to the futures price. From (1.7) the relation between the futures price and the implied Eurodollar Futures rate $\tilde{r}_H$ is given by

$$H(t,u^*,\tau^*) =: 1-(\tau^*-u^*)\tilde{r}_H(t,u^*,\tau^*).$$

Therefore the Eurodollar option corresponds to an option on the futures price. In a similar way, we can now compute the arbitrage price of a futures style option on a futures contract with exercise time $s \geq t$, $s < \tau^*$ and strike K:

$$\begin{aligned}
& E_{P^*}\left[[H(s,u^*,\tau^*)-K]^+|\mathbb{F}_t\right] \qquad (3.18)\\
= & E_{P^*}\left[H(s,u^*,\tau^*)\cdot\left[1-K\cdot H(s,u^*,\tau^*)^{-1}\right]^+|\mathbb{F}_t\right]\\
= & H(t,u^*,\tau^*)\cdot E_{Q_H^{u^*,\tau^*}}\left[[1-K(1+(\tau^*-u^*)\cdot r_H(s,u^*,\tau^*)]^+|\mathbb{F}_t\right]\\
= & H(t,u^*,\tau^*)\cdot[(1-K)\cdot N(e_1)-(\tau^*-u^*)\cdot K\cdot r_H(t,u^*,\tau^*)\cdot N(e_2)]\\
= & H(t,u^*,\tau^*)(1-K)N(e_1)-K(1-H(t,u^*,\tau^*))N(e_2),
\end{aligned}$$

with

$$e_{1/2} = \frac{\ln\left(\frac{(1-K)H(t,u^*,\tau^*)}{K(1-H(t,u^*,\tau^*))}\right) \pm \frac{1}{2}g^2}{g} \quad \text{and} \quad g^2 := \int_t^s ||\gamma_H(\theta,u^*,\tau^*)||^2 d\theta.$$

The structure of this formula coincides with the arbitrage price of an option on a zero coupon bond in the LIBOR Market Model.

4 Conclusion

The approach taken in this paper is the definition of a no-arbitrage model, based on futures prices, for the term structure of interest rates. At a first view this approach could seem to be mainly of theoretical interest. This intuition turns out to be wrong. Similar to option prices, also futures prices add information to the structure of the model. This additional information implies that the expected spot rate and the volatility structure of the futures prices are closely related to the initial futures prices. Two main consequences can be drawn.

First, the expected spot rate is through equation (2.14) related to the initial futures intensity under the martingale measure. This opens a new and, to our knowledge, so far unconsidered way to value the accuracy of an assumed model specification for the term structure of interest rates. Since this property is independent of the volatility structure of the futures prices, it can be used to justify a specific volatility structure even if the structure can only be analyzed in numerical ways.

Second, the main result in Proposition 2.2 implies that the volatility structure itself is related to the initial futures intensity. This is of importance in two ways. For short times to maturity an approximation of futures prices by a sufficiently smooth function can be used to fix the short end of the term structure of volatility. It means that, in addition to an estimation on the basis of a time series approach, the implied term structure of volatility can be expressed by using futures prices. Furthermore, a majority of the volatility structures applied in practice assume a deterministic specification. In this case our results imply that the initial futures prices can be used to completely determine this function. As an example this is shown for a multidimensional Vasicek model. In the same way this can be shown for other model specifications. This again allows us to relate a specific structure for the term structure of volatilities to market information. Furthermore, Example 2.3 seems to indicate that the funtional form of the initial intensity is related to the number of factors entering the term structure of interest rates. More precisely, suppose that the initial futures intensity can be approximated by a sum of basis functions. In this case the number of basis functions is equal to the number of factors and each basis function determines the structure of the corresponding projection of the volatility vector. In addition to these features, the paper adds a new measure transformation to the existing martingale results. Assuming sufficient regularity, the no-arbitrage condition implies that discounted spot and futures prices are martingales under the martingale measure, and that forward prices and forward rates are martingales under the appropriate forward risk adjusted measure. It is shown that nominal futures rates are martingales under the appropriate futures risk adjusted measure. As for the case of standard options this property is central to compute the no-arbitrage value of futures style options. Parallel to option pricing techniques for non-futures style options, this measure transformation can be used to efficiently compute the option value of futures style options.

To show this, we consider a LIBOR market structure. Again two results are derived. First, assuming log-normality yields the same structural problem already known in the LIBOR market situation. Under no-arbitrage it is not possible to impose log-normality on all nominal futures rate processes. In contrast to the LIBOR market this result is not derived by a simple duplication argument in connection with the instability of the log-normal distribution with respect to summation. In the futures market case this assumption is not compatible with the no-arbitrage restrictions on the volatility given by Proposition 2.2. On the other hand these restrictions do not prevent us from assuming log-normality for a limited number of processes, i.e. as in the LIBOR market case, the term structure of volatilities is not completely specified. In this case future style options on futures prices and futures rates can be calculated in closed form. In particular this yields a new pricing formula for a Eurodollar futures option. The interpretation of this formula is closely related to Blacks formula for the caplet and floorlet. Instead of the forward rate volatility the futures price volatility enters the formula. Furthermore, the margin system implies an additional option part which intuitively arises from the payment in advance property of a futures style option.

From an applied point of view the paper focuses on two aspects: The relationship between futures prices and the term structure of volatilities and the necessity to use observable data to further develop the no-arbitrage theory of the term structure of interest rates.

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